Lecture 21 (11/18/2019)

The statements $V_1 \oplus V_2 \oplus \dots \oplus V_n = V$ means two things:

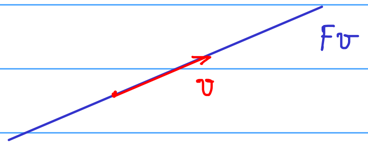
(1) The sum $V_1 + V_2 + \dots + V_n$ is a direct sum,

(2) $V_1 + V_2 + \dots + V_n = V$.

Ex:

Let V be a vector space over a field of numbers F, which could be $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. For each $v \in V$, we use the notation

$$Fv := \{cv : c \in F\} = \text{span}\{v\}.$$



If $v \neq 0$ then F is a 1-dimensional subspace with basis $\{v\}$.

Suppose $B = \{v_1, v_2, \dots, v_n\}$ is a basis of V. Then

$$V = Fv_1 \oplus Fv_2 \oplus \dots \oplus Fv_n. \qquad (*)$$

In other words, V is "decomposed" into n one-dimensional subspaces. Why is $(*)$ true?

We need to show two things:

(1) The sum $Fv_1 + \dots + Fv_n$ is a direct sum.

(2) $Fv_1 + \dots + Fv_n = V$.

To show (1), we take a basis of each subspace Fv_i.

Choose $B_i = \{v_i\}$. The concatenation is

$$B_1 \sqcup B_2 \sqcup \dots \sqcup B_n = \{v_1, v_2, \dots, v_n\}.$$

This set is linearly independent because it is a basis of V. Therefore, (1) is true.

To show (2), we notice that $Fv_1 + \dots + Fv_n$ is a subspace of V. Because $Fv_1 + \dots + Fv_n$ is a direct sum,

$$\dim(Fv_1 + \dots + Fv_n) = \underbrace{\dim Fv_1}_{1} + \dots + \underbrace{\dim Fv_n}_{1}$$

$$= n = \dim V.$$

Therefore, $Fv_1 + \cdots + Fv_n = V$.

<u>Ex</u> Consider the following subspaces of $M_{2\times 2}(\mathbb{R})$

$$V_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+d = b+c = 0 \right\}$$

$$V_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+c = b = d = 0 \right\}$$

$$V_3 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b = c = d = 0 \right\}$$

Show that $V_1 \oplus V_2 \oplus V_3 = M_{2\times 2}(\mathbb{R})$.

Our strategy is to convert this problem into a problem in $\mathbb{R}^4$ by using coordinates. Consider the standard basis of $M_{2\times 2}(\mathbb{R})$:

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Each vector $v = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $M_{2\times 2}(\mathbb{R})$ corresponds to

a vector $[v]_B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ of $\mathbb{R}^4$.

V_1 has a basis $B_1 = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{u_2} \right\}.$

V_2 has a basis $B_2 = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}}_{u_3} \right\}$

V_3 has a basis $B_3 = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{u_4} \right\}$

We convert the problem on $M_{2\times 2}(\mathbb{R})$ to a problem on $\mathbb{R}^4$ as follows.

V_1 corresponds to a subspace V_1' of $\mathbb{R}^4$ with basis

$$B_1' = \{\underbrace{(1,0,0,-1)}_{u_1'}, \underbrace{(0,1,-1,0)}_{u_2'}\}$$

V_2 corresponds to a subspace V_2' of $\mathbb{R}^4$ with basis

$$B_2' = \{\underbrace{(1,0,-1,0)}_{u_3'}\}$$

V_3 corresponds to a subspace V_3' of $\mathbb{R}^4$ with basis

$$B_3' = \{\underbrace{(1,0,0,0)}_{u_4'}\}$$

We want to show $V_1' \oplus V_2' \oplus V_3' = \mathbb{R}^4$.

This means that we need to show two things:

(1) $V_1' + V_2' + V_3'$ is a direct sum,

(2) $V_1' + V_2' + V_3' = \mathbb{R}^4$.

Show (1):

Concatenate the bases: $\begin{bmatrix} | & | & | & | \\ u_1' & u_2' & u_3' & u_4' \\ | & | & | & | \end{bmatrix}$ with $\underbrace{u_1' \; u_2'}_{B_1'}$, $\underbrace{u_3'}_{B_2'}$, $\underbrace{u_4'}_{B_3'}$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} I_4$$

Thus, $B_1' \sqcup B_2' \sqcup B_3'$ is linearly independent.

Show (2):

We know that $V_1' + V_2' + V_3'$ is a subspace of $\mathbb{R}^4$ and

$$\dim(V_1'+V_2'+V_3') = \dim V_1' + \dim V_2' + \dim V_3'$$

(due to direct sum)

$$= 2 + 1 + 1$$

$$= 4$$

$$= \dim \mathbb{R}^4.$$

Therefore, $V_1' + V_2' + V_3' = \mathbb{R}^4$.

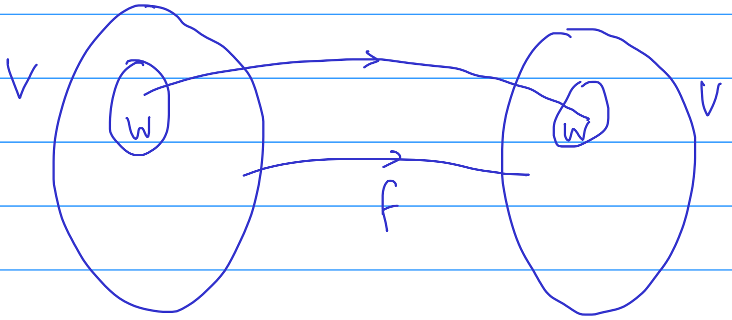
* Invariant subspaces:

Recall the definition: let $f: V \to V$ be a linear map. Note that f goes from V to itself. A subspace $W \subset V$ is called invariant under f if $f(W) \subset W$.

Note:

$$f(W) \overset{\text{def}}{=} \{f(x) : x \in W\}$$

(the set of the images of vectors in W under f.)



Intuitively, if W is invariant under f then f can be "localized" to W.

To show $f(W) \subset W$, one starts by writting:

"Take $x \in W$. We want to show $f(x) \in W$."

See an example on the worksheet.