Lecture 20 (11/15/2019)

* There are several ways to check if $U+V$ is a direct sum. Below are some (usually) quick ways:

1) Check if $U \cap V = \{0\}$. For this method, one can start by writing: "Let $v \in U \cap V$. We check if (or show that) $v = 0$."

2) Find a basis B_1 of U, basis B_2 of V. Then check if $\underbrace{B_1 \amalg B_2}_{}$ is linearly independent.
Concatenation of B_1 and B_2
Note that concatenation is almost the same as union, except that we don't remove the repeated vector. For example,

$$B_1 = \{(0,1,0), (0,0,1)\}$$
$$B_2 = \{(1,0,0), (0,0,1)\}$$

Then

$$B_1 \cup B_2 = \{(0,1,0), (0,0,1), (1,0,0)\} \quad \text{(union)}$$
$$B_1 \amalg B_2 = \{(0,1,0), (0,0,1), (0,0,1), (1,0,0)\} \quad \text{(concatenation)}$$

In this example, $B_1 \cup B_2$ is linearly independent, but $B_1 \amalg B_2$ is not.

3) If $U, V \subset W$, and $\dim U + \dim V > \dim W$ then $U+V$ is <u>not</u> a direct sum.
Why?
We know that $U+V$ is the smallest vector space that contains both U and V. Since W contains both U and V, it must contain $U+V$. Then

$$\dim(U+V) \leq \dim W < \dim U + \dim V.$$

Thus, $U+V$ is not a direct sum.

<u>Ex</u>: Consider two subspaces of $\mathbb{R}^3$:

$U = \{(x, y, z) : x + y = 0\}$

$V = \{(x, y, z) : z + y = 0\}$

Show that $U + V$ is not a direct sum.

One can find the intersection $U \cap V$ can realize that it is not $\{0\}$. Geometrically, U and V are planes in $\mathbb{R}^3$ intersecting each other at a line. Here we will try to use Method 3:

U is 2-dimensional (3 free variables − 1 constraint)

V is 2-dimensional (3 free variables − 1 constraint)

$U, V \subset \mathbb{R}^3$, which is 3-dimensional.

Because

$$\dim U + \dim V = 4 > \dim \mathbb{R}^3,$$

the sum $U + V$ is not a direct sum.

* There are several ways to check if $V_1 + V_2 + \cdots + V_n$ is a direct sum. Below are some (usually) quick ways:

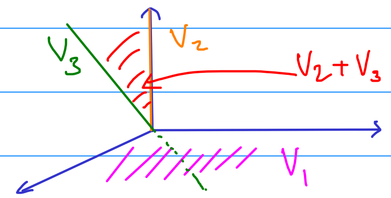
1) Find a basis B_1 of $V_1, \ldots, B_n$ of V_n. Then check if (or show that) the concatenation $B_1 \sqcup B_2 \sqcup \cdots \sqcup B_n$ is linearly independent.

2) Let $v_1 + v_2 + \cdots + v_n = 0$ for some $v_1 \in V_1, \ldots, v_n \in V_n$. Check if $v_1 = v_2 = \cdots = v_n = 0$.

3) If $V_1, \ldots, V_n \subset W$ and $\dim V_1 + \cdots + \dim V_n > \dim W$ then $V_1 + \cdots + V_n$ is <u>not</u> a direct sum.

Warning: for $n > 2$, the fact that $V_1 + \cdots + V_n$ is a direct sum is <u>not</u> equivalent to the fact that the intersection $V_1 \cap V_2 \cap \cdots \cap V_n$ is equal to $\{0\}$.

Ex:



$V_1 =$ xy-plane

$V_2 =$ z-axis

$V_3 =$ line (ℓ)

We see that $V_1 \cap V_2 = V_2 \cap V_3 = V_1 \cap V_3 = V_1 \cap V_2 \cap V_3 = \{0\}$.
However, $V_1 + V_2 + V_3$ is not a direct sum because $(V_2+V_3) \cap V_1 \neq \{0\}$.
This intersection is a line.

Recall that for $V_1 + \dots + V_n$ to be a direct sum, one must have $V_i \cap (V_1 + \dots + V_{i-1} + V_{i+1} + \dots + V_n) = \{0\}$ for all $i = 1, 2, \dots, n$.

Ex:

$$V_1 = \{(x,y,z) : x+y = z = 0\}$$
$$V_2 = \{(x,y,z) : x+2y = y+z = 0\}$$
$$V_3 = \{(x,y,z) : x = 2y = 3z\}.$$

Show that $V_1 \oplus V_2 \oplus V_3 = \mathbb{R}^3$.

The problem asks us to show 2 things:

(1) $V_1 + V_2 + V_3$ is a direct sum.

(2) $V_1 + V_2 + V_3 = \mathbb{R}^3$

- Let us show (1):

We will find a basis for each V_1, V_2, V_3.

$V_1 = \{(x, -x, 0) : x \in \mathbb{R}\} = \operatorname{span}\{(1,-1,0)\}$

has basis $B_1 = \{(1,-1,0)\}$.

$V_2 = \{(x, -x/2, x/2) : x \in \mathbb{R}\} = \operatorname{span}\{(1, -1/2, 1/2)\}$

has basis $B_2 = \{(1, -1/2, 1/2)\}$.

$V_3 = \{(x, \frac{x}{2}, \frac{x}{3}) : x \in \mathbb{R}\} = \operatorname{span}\{(1, \frac{1}{2}, \frac{1}{3})\}$

has basis $B_3 = \{(1, \frac{1}{2}, \frac{1}{3})\}$

Concatenation of bases:

$B_1 \sqcup B_2 \sqcup B_3 = \{(1,-1,0), (1, -1/2, 1/2), (1, 1/2, 1/3)\}$.

These vectors are linearly independent because

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & -1/2 & 1/2 \\ 1 & 1/2 & 1/3 \end{vmatrix} \neq 0.$$

Therefore, $V_1 + V_2 + V_3$ is a direct sum.

- Show (2):

We know that $V_1 + V_2 + V_3$ is a subspace of $\mathbb{R}^3$.
we also know from above that $\dim(V_1 + V_2 + V_3) = 3$.
Therefore, $V_1 + V_2 + V_3 = \mathbb{R}^3$.

Ex:

Let $A \in M_{n \times n}(\mathbb{R})$. Consider the following sets:

$$V_1 = \{v \in \mathbb{R}^n : Av = v\}$$
$$V_2 = \{v \in \mathbb{R}^n : Av = 2v\}$$
$$V_3 = \{v \in \mathbb{R}^n : Av = 3v\}.$$

Show that $V_1 + V_2 + V_3$ is a direct sum.

In this problem, it is hard to use the first method (i.e. find a basis for each V_1, V_2, V_3) because A is quite a general matrix. We will try the second method:

Let $v_1 \in V_1$, $v_2 \in V_2$, $v_3 \in V_3$ be such that $v_1 + v_2 + v_3 = 0$.
We will show that $v_1 = v_2 = v_3 = 0$.

Multiplying both sides of the equation by A:

$$A(v_1 + v_2 + v_3) = A0 = 0$$

which is equivalent to

$$Av_1 + Av_2 + Av_3 = 0$$

which is equivalent to

$$v_1 + 2v_2 + 3v_3 = 0.$$

Multiplying this equation by A, we get

$$A(v_1 + 2v_2 + 3v_3) = A0 = 0$$

which is equivalent to

$$Av_1 + 2Av_2 + 3Av_3 = 0$$

which is equivalent to

$$v_1 + 4v_2 + 9v_3 = 0.$$

We have got three equations:

$$\begin{cases} v_1 + v_2 + v_3 = 0 \\ v_1 + 2v_2 + 3v_3 = 0 \\ v_1 + 4v_2 + 9v_3 = 0 \end{cases}$$

Note that v_1, v_2, v_3 are vectors, not numbers. However, one can use familiar methods to solve for v_1, v_2, v_3, for example by substitution and elimination. If v_1, v_2, v_3 were numbers, we can write the system in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now that v_1, v_2, v_3 are vectors, we only need to adjust this form slightly to make it correct:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} — v_1 — \\ — v_2 — \\ — v_3 — \end{bmatrix}}_{3\times n} = \underbrace{\begin{bmatrix} — 0 — \\ — 0 — \\ — 0 — \end{bmatrix}}_{3\times n}$$

Multiply both sides by the inverse of the constant matrix:

$$\begin{bmatrix} — v_1 — \\ — v_2 — \\ — v_3 — \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}^{-1} \begin{bmatrix} — 0 — \\ — 0 — \\ — 0 — \end{bmatrix} = 0$$

Therefore, $v_1 = v_2 = v_3 = 0$. We conclude that $V_1 + V_2 + V_3$ is a direct sum.