Lecture 2 (9/27/2019)

- Vector space $\mathbb{R}^n$: one can talk about addition of two vectors, scaling of a vector, linear combination, linear independence, spanning set (or generating set), basis.

To study these topics, one only focuses on a few features of $\mathbb{R}^n$: being able to add and scale vectors. One doesn't need to care about how to multiply or divide two vectors. In fact, with $V = \mathbb{R}^n$ and $F = \mathbb{R}$ we observe that the following properties are satisfied.

(A) Addition:

(A0) There is an "intrinsic" addition operator:

If $x, y \in V$ then $x + y \in V$.

(A1) $x + y = y + x$ (commutative)

(A2) $(x+y)+z = x+(y+z)$ (associative)

(A3) $x + \text{"}0\text{"} = x$ (having an additive identity)

(A4) $x + \text{"}-x\text{"} = 0$ (having an additive inverse)

(S) Scaling (or scalar multiplication) by factors in F:

(S0) There is an "external" multiplication with numbers:

If $x \in V$ and $c \in F$ then $cx \in V$.

(S1) $c(dx) = (cd)x$ (associative)

scaling product of numbers scaling

(S2) $1x = x$ (scaling by 1)

The set of numbers can be $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ (called field).

(I) Interaction between addition and scaling:

(I1) $c(x+y) = cx + cy$ ("first add, then scale" is the same as "first scale, then add")

(I2) $(c+d)x = cx + dx$

These two identities are called distribution property.

* A structure consisting of a set V, a set of numbers $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, addition, scalar multiplication that satisfies (A), (S), (I) is called a vector space over F.
* Each of properties (A0), (A1), ..., (I2) is called an axiom.

One can think of "vector space" as a recipe in a cookbook. A recipe has two parts: ingredients and directions.

Ingredients:
- set V
- field of numbers F
- addition
- scaling

Directions: (A), (S), (I) are satisfied.

<u>Ex 1</u>: Only assuming (A0), (A1), (A2), is it true that

$$(u+v)+w = v+(u+w) \text{ for all } u, v, w \in V \text{ ?}$$

Yes. The reason is as follows.

$$\begin{aligned}(u+v)+w &= (v+u)+w && \text{by (A1)}\\ &= v+(u+w) && \text{by (A2)}\end{aligned}$$

<u>Ex 2</u>: Only assuming group (A), is it true that each $v \in V$ has exactly one additive inverse?

Yes. Let $v \in V$. Suppose u and w are inverses of v. We want to show that $u = w$.

The only thing we know about u and w is

$$u + v = 0$$

$$w + v = 0$$

Add both sides of the first equation to w:

$$(u+v)+w = 0+w$$

By (A3), RHS = w

By (A2), LHS $= u + (v+w)$

$= u + 0$

$= u$ by (A3)

Therefore, $u = w$.

<u>Ex 3</u>: Assuming only (A) and (S), is it true that $(-1)v = -v$ for all $v \in V$?

Well, let's analyse. The equation

$$(-1)v = -v \qquad (1)$$

is equivalent to

$$(-1)v + v = (-v) + v. \qquad (2)$$

In other words, if (1) is true then (2) is true. If (2) is true, then (1) is true (by adding $-v$ on both sides).

We see that RHS (2) $= 0$ by (A4).

LHS (2) $= (-1)v + 1v$

We cannot write

$$(-1)v + 1v = (-1+1)v$$

because this would need (I). The property (I2) is included in the definition of vector space so that the identity (1) is true.

* Comments:

- If $V = \mathbb{R}^n$ and $F = \mathbb{R}$, the answer to Ex 1,2,3 is obviously yes. However, for abstract V, the answer is not obvious and requires proof as we showed above.
- Why do we need to care about such an abstract definition? Should we just be content with $\mathbb{R}^n$?

Last time we considered an example in which the idea of eigenvectors yields an important result in mathematics and signal processing – Fourier series. Each "vector" was a function (a much more complicated object than a vector in $\mathbb{R}^n$).

The abstract notion of vector space makes $\mathbb{R}^n$ more useful; many results on $\mathbb{R}^n$ can be used for other sets, for example the set of functions or matrices.