Lecture 19 (11/13/2019)

Recall: the sum $U+V$ is called a direct sum if $U \cap V = \{0\}$.

The following statements are equivalent:

(1) $U \cap V = \{0\}$.

(2) $B_1 \cup B_2$ is a basis of $U+V$, where B_1 is a basis of U and B_2 is a basis of V.

(3) $\dim(U+V) = \dim U + \dim V$.

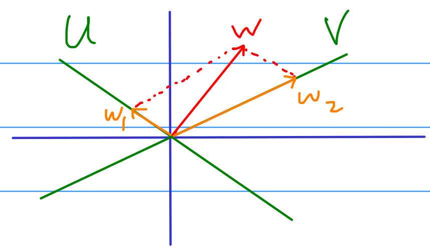
(4) Each element w in $U+V$ can be written uniquely as $w = u+v$ where $u \in U$ and $v \in V$.
In other words, if $u+v = u'+v'$ for some $u, u' \in U$ and $v, v' \in V$ then $u=u'$ and $v=v'$.

(5) If $u+v=0$ for some $u \in U$ and $v \in V$ then $u=v=0$.

Methods (1), (2), (5) are usually easy to use in solving problems.

Ex: in $\mathbb{R}^2$, consider two different lines U and V. Then

$$\mathbb{R}^2 = U \oplus V$$



Each vector $w \in \mathbb{R}^2$ can be expressed as $w = w_1 + w_2$ where $w_1 \in U$ and $w_2 \in V$. Such representation is unique.

* Sum of many vector spaces:

Let $V_1, V_2, \dots, V_n$ be vector spaces. Then $V_1+V_2+\dots+V_n$ is the set defined by

$$V_1+V_2+\dots+V_n = \{v_1+v_2+\dots+v_n : v_1 \in V_1, v_2 \in V_2, \dots, v_n \in V_n\}. \quad (*)$$

This is the smallest vector space that contains all $V_1, V_2, \dots, V_n$.

<u>Definition:</u>

The sum $V_1 + \dots + V_n$ is called a direct sum, denoted by $V_1 \oplus V_2 \oplus \dots \oplus V_n$, if for any basis B_1 of V_1, B_2 of $V_2, \dots$, B_n of V_n, the union $B_1 \cup B_2 \cup \dots \cup B_n$ is a basis of $V_1 + \dots + V_n$.

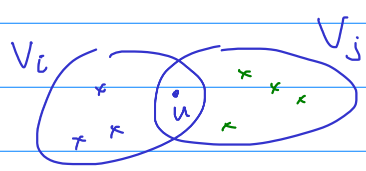
* <u>Note:</u> In the definition $(*)$ of $V_1 + \dots + V_n$, we can see that the set $B_1 \cup B_2 \cup \dots \cup B_n$ spans $V_1 + \dots + V_n$. The essence of the definition of direct sum is that:

$V_1 + \dots + V_n$ is a direct sum if and only if the concatenation of $B_1, B_2, \dots, B_n$ is linearly independent.

See Problem 1 on the worksheet for example.

In this problem, $V_2 \cap V_4 \neq \{0\}$ because $(0,1,0,0) \in V_2 \cap V_4$.

In general, if $V_i \cap V_j \neq \{0\}$ for some $i \neq j$ then $V_1 + V_2 + \dots + V_n$ is not a direct sum.



Why so? Let u be a nonzero vector that belongs to both V_i and V_j. One can add more vectors (selected in V_i) to the set $\{u\}$ to form a basis B_i for V_i. Similarly, one can add more vectors (selected in V_j) to the set $\{u\}$ to form a basis B_j for V_j.

When we concatenate the bases $B_1, B_2, \dots, B_n$ we get a list of vectors

$$\underbrace{- - -}_{B_1} \; \dots \; \underbrace{- - -, u, - - -}_{B_i} \; \underbrace{- - -, u, - -}_{B_j} \; \dots \; \underbrace{- - -}_{B_n}$$

These vectors are not linearly independent because vector u appears twice.

Therefore, if there are two among $V_1, V_2, \dots, V_n$ that have nontrivial intersection then $V_1 + \dots + V_n$ is not a direct sum.

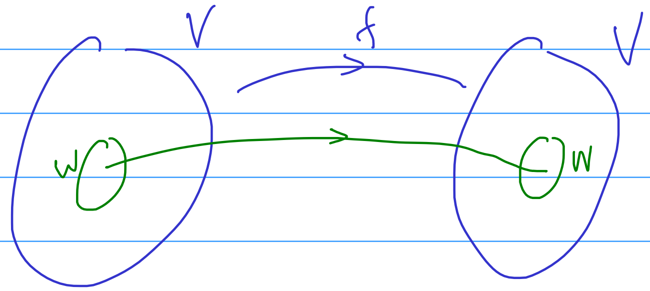
The following statements are equivalent:

1) $V_1 + V_2 + \dots + V_n$ is a direct sum.
2) The concatenation of $B_1, \dots, B_n$ (where B_i is a basis of V_i) is linearly independent.
3) $\dim(V_1 + \dots + V_n) = \dim V_1 + \dots + \dim V_n$
4) For every $1 \le i \le n$,
$$V_i \cap (V_1 + \dots + V_{i-1} + V_{i+1} + \dots + V_n) = \{0\}.$$
5) Each $v \in V_1 + \dots + V_n$ can be written uniquely as
$$v = v_1 + v_2 + \dots + v_n$$
where $v_i \in V_i$. In other words, if $v_1 + \dots + v_n = v_1' + \dots + v_n'$ for $v_i, v_i' \in V_i$ then $v_1 = v_1'$, $v_2 = v_2'$, $\dots$, $v_n = v_n'$.
6) If $v_1 + v_2 + \dots + v_n = 0$ for some $v_1 \in V_1, \dots, v_n \in V_n$ then $v_1 = v_2 = \dots = v_n = 0$.

We introduce the next concept as a final preparation for spectral theory:

Let $f: V \to V$ be a linear map. (Note that the domain is the same as the target set.) A linear map from a vector space to itself is called an endomorphism.

A subspace W of V is said to be invariant under f if $f(W) \subset W$. In other word, f maps a vector of W to a vector in W.



See Problem 2 on the worksheet for an example.