Lecture 18 (11/08/2019)

Last time, we discussed how to find a basis for $U+V$, where U and V are subspaces of $\mathbb{R}^k$ (in general, F^k where $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$). See problem 2 on the last worksheet for an example.

To summarize the method: first, we find a basis of U, called $B_1 = \{u_1, u_2, \dots, u_n\}$. Then find a basis of V, called $B_2 = \{v_1, \dots, v_m\}$. At this point, we know that

$$U+V = \text{span}\{u_1, \dots, u_n, v_1, \dots, v_m\} = Col(A)$$

where A is the matrix

$$A = \begin{bmatrix} | & & | & | & & | \\ u_1 & \cdots & u_n & v_1 & \cdots & v_m \\ | & & | & | & & | \end{bmatrix}$$

To find a basis of $Col(A)$, we reduce A into reduced row echelon form (RREF):

$$A \xrightarrow{RREF} \begin{bmatrix} & & \\ & \cdots\cdots & \\ & & \end{bmatrix}$$

pivot columns

tell us which column of A to take to form a basis of $Col(A)$.

* What if U, V are subspaces of a general vector space W (other than the space $\mathbb{Q}^n, \mathbb{R}^n, \mathbb{C}^n$)? Consider an example:

$$W = P_3(\mathbb{R})$$
$$U = \text{span}\{x, 1\}$$
$$V = \text{span}\{x^2+1, x+2\}$$

Find a basis of $U+V$.

One can translate this problem into a problem in $\mathbb{R}^n$ as follows. We see that W is isomorphic to $\mathbb{R}^4$ in the following way: each vector $u \in P_3$ is associated with the coordinate vector $[u]_B \in \mathbb{R}^4$ where B is the standard basis of P_3:

$$B = \{x^3, x^2, x, 1\}.$$

Each $u = ax^3 + bx^2 + cx + d$ in P_3 corresponds to a coordinate vector

$$[u]_B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4.$$

The map $\phi: P_3 \to \mathbb{R}^4$ given by $\phi(u) = [u]_B$ is a linear isomorphism.

Why? Note that $\dim P_3 = \dim \mathbb{R}^4 = 4$ and ϕ is onto.

Therefore, the problem on P_3 can be translated into a problem on $\mathbb{R}^4$. Instead of working with polynomials, we work with their coordinate vectors. Then translate back to polynomials at the end.

Return to the problem:

$$[\underbrace{x}_{u_1}]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad [\underbrace{x^2+1}_{v_1}]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$[\underbrace{1}_{u_2}]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad [\underbrace{x+2}_{v_2}]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

Subspace U of P_3 corresponds to subspace U' of $\mathbb{R}^4$:

$$U' = \text{span}\left\{ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{u_1'}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{u_2'} \right\}$$

Subspace V of P_3 corresponds to subspace V' of $\mathbb{R}^4$:

$$V' = \text{span}\left\{ \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{v_1'}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}}_{v_2'} \right\}$$

$U+V$ corresponds to $U'+V'$. To find a basis of $U'+V'$, we find a basis for the column space of matrix

$$A = \begin{bmatrix} | & | & | & | \\ u_1' & u_2' & v_1' & v_2' \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

One can use Matlab to find RREF of this matrix: type the commands

```
>> A= [0 0 0 0; 0 0 1 0; 1 0 0 1; 0 1 1 2]
>> rref(A)
```

which give

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$

pivot columns

The 1^{st}, 2^{nd}, 3^{rd} columns of A form a basis for $U'+V'$.

$$B' = \{u_1', u_2', v_1'\}$$

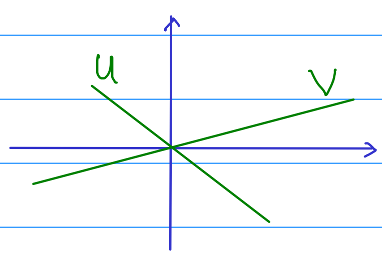
Thus, $B = \{u_1, u_2, v_1\}$ form a basis of $U+V$.

$$B = \{x, 1, x^2+1\}$$

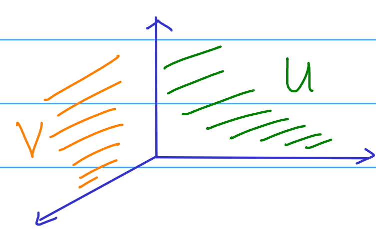
Direct sum

If vector spaces U and V has the minimal intersection, i.e. $U \cap V = \{0\}$ then the sum $U+V$ is said to be a direct sum, denoted by $U \oplus V$.

If $U+V$ is a direct sum then $B_1 \cup B_2$ is a basis of $U+V$ (i.e. there is no "redundant" vector in $B_1 \cup B_2$.)



$U+V$ is a direct sum



$U+V$ is not a direct sum

* Theorem:

$U+V$ is a direct sum if and only if

$$\dim(U+V) = \dim U + \dim V.$$

Why?

Let B_1 and B_2 be bases of U and V respectively.

If $U+V$ is a direct sum then $B_1 \cup B_2$ is a basis of $U+V$.

$$\begin{aligned} \dim(U+V) &= \#(B_1 \cup B_2) \\ &= \# B_1 + \# B_2 \quad (B_1 \text{ and } B_2 \text{ are disjoint}) \\ &= \dim U + \dim V \end{aligned}$$

Conversely, if $\dim(U+V) = \dim U + \dim V$ then

$$\underbrace{\#\text{ vector in a basis of } U+V}_{=\ \#\text{ linearly independent vectors in } B_1 \cup B_2} = \# B_1 + \# B_2$$

This is possible only if vectors in $B_1 \cup B_2$ are linearly ind.

Thus, U and V has no common vectors other than 0.