Lecture 17 (11/06/2019)

Recall: $U+V = \{u+v : u \in U, v \in V\}$

This is the smallest vector space that contains both U and V.

Indeed, for any vector space W that contains U and V, we have

$$u + v \in W \qquad \forall u \in U, v \in V$$

because W is closed under addition. Therefore, $U+V \subset W$.

* How to show that a set A is equal to a set B?

We show 2 things:

$$\begin{cases} A \subset B & (1) \\ B \subset A & (2) \end{cases}$$

To show (1), take an element $x \in A$. Show that $x \in B$.

To show (2), take an element $x \in B$. Show that $x \in A$.

* Suppose U and V are vector spaces. To show that $U \subset V$, we only need to show that V contains a basis of U.

Why so? Let $B = \{u_1, u_2, \dots, u_n\}$ be a basis of U that is contained in V. We want to show $U \subset V$.

To do so, we take any vector $u \in U$ and show that $u \in V$.

Because $u \in U$, u is a linear combination of vectors in B:

$$u = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$$

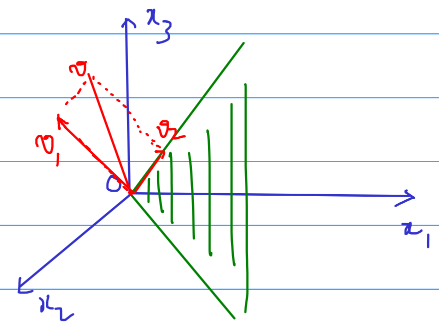
This belongs to V because $u_1, u_2, \dots, u_n \in V$ and V is closed under addition and scaling.

Ex:

$$U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0\}$$
$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 + x_3\}$$

Show that $U + V = \mathbb{R}^3$.

Before doing any rigorous proof, one can convince himself that $U+V = \mathbb{R}^3$ by visualizing U and V:



$U = x_2x_3$- plane

V = the shaded plane

Any vector $v \in \mathbb{R}^3$ can be written as $v = v_1 + v_2$ where $v_1 \in U$ and $v_2 \in V$.

We know that $U + V \subset \mathbb{R}^3$. We only need to show $\mathbb{R}^3 \subset U + V$. To do so, we need to find a basis of $\mathbb{R}^3$ that is contained in $U+V$.

Let's find a basis of U and a basis of V.

$$\begin{aligned} U &= \{(x_1, x_2, x_3) : x_1 = 0\} \\ &= \{(0, x_2, x_3) : x_2, x_3 \in \mathbb{R}\} \\ &= \{x_2(0,1,0) + x_3(0,0,1) : x_2, x_3 \in \mathbb{R}\} \\ &= \operatorname{span}\{(0,1,0), (0,0,1)\} \end{aligned}$$

After checking linear independence, we conclude that $u_1 = (0,1,0)$ and $u_2 = (0,0,1)$ form a basis of U.

$$\begin{aligned} V &= \{(x_1, x_2, x_3) : x_1 = x_2 + x_3\} \\ &= \{(x_2 + x_3, x_2, x_3) : x_2, x_3 \in \mathbb{R}\} \\ &= \{x_2(1,1,0) + x_3(1,0,1) : x_2, x_3 \in \mathbb{R}\} \\ &= \operatorname{span}\{(1,1,0), (1,0,1)\} \end{aligned}$$

After checking linear independence, we conclude that $v_1 = (1,1,0)$ and $v_2 = (1,0,1)$ form a basis of V.

We know that $U+V$ contains u_1, u_2, v_1, v_2. It is easy to check that u_1, u_2, v_1 are linearly independent, for example by checking that the determinant is nonzero.

$$\begin{vmatrix} | & | & | \\ u_1 & u_2 & v_1 \\ | & | & | \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \neq 0$$

* In general, how to find a basis for the sum $U+V$?

Let us assume U and V are subspaces of $\mathbb{R}^n$.

The overall idea is as follows:

- First, find a basis B_1 for U, say $B_1 = \{u_1, \dots, u_n\}$.
- Secondly, find a basis B_2 for V, say $B_2 = \{v_1, \dots, v_m\}$.

Because B_1 spans U and B_2 spans V, the union

$$B_1 \cup B_2 = \{u_1, \dots, u_n, v_1, \dots, v_m\}$$

spans the space $U+V$.

- Thirdly, we try to eleminate the redundant vectors in the set $B_1 \cup B_2$. What remains is a basis of $U+V$.

There is an algorithm for this step. We are dealing with the problem of extracting a basis from the spanning set $\{u_1, \dots, u_n, v_1, \dots, v_m\}$.

In Math 341, we learned how to do so: this is the problem of find a basis for the column space of matrix

$$A = \begin{bmatrix} | & | & & | & | & & | \\ u_1 & u_2 & \cdots & u_n & v_1 & \cdots & v_m \\ | & | & & | & | & & | \end{bmatrix}$$

$$\mathrm{Col}(A) = \mathrm{span}\{u_1, u_2, \ldots, u_n, v_1, \ldots, v_m\} = U + V.$$

$$A \xrightarrow{RREF} A' = \begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix}$$

the pivot columns in A'
tell us which column in A to
take to form a basis for $\mathrm{Col}(A)$.

Ex:

In the previous example,

$B_1 = \{u_1, u_2\}$,

$B_2 = \{v_1, v_2\}$,

$$A = \begin{bmatrix} | & | & | & | \\ u_1 & u_2 & v_1 & v_2 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

pivot columns

Take the 1^{st}, 2^{nd}, 3^{rd} columns of A.

Conclusion: $\{u_1, u_2, v_1\}$ is a basis of $U + V$.

See another example on worksheet.