Lecture 16 (11/4/2019)

We have learned vector spaces. A vector space is a set with certain structures (two operations and a base field of numbers). We have also learned linear maps. A linear map is a map between two vector spaces that is compatible with the structure of vector spaces (additive and scalar multiplicative).

Each n-dimensional vector space V over $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ is associated with a linear map $f: V \longrightarrow F^n$ given by

$$f(v) = [v]_B.$$

Here B is a basis of V. This map is an isomorphism, i.e. linear + one-to-one + onto, making V "equivalent" to F^n. In other words, the vector space F^n captures all features of V as a vector space.

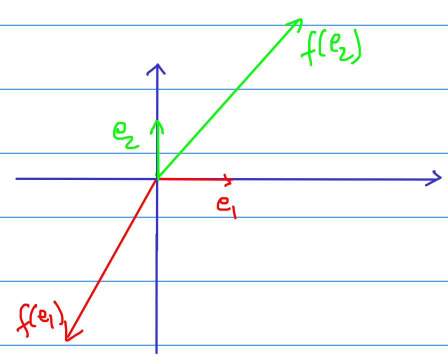
Our next theme is <u>decomposition of a vector space</u>. Let's consider the following example. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x_1, x_2) = (-4x_1 + 3x_2, -6x_1 + 5x_2).$$

This is a linear map with matrix representation (in standard basis of $\mathbb{R}^2$):

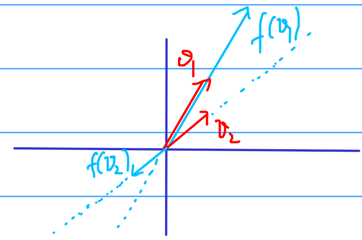
$$A = \begin{bmatrix} | & | \\ [f(e_1)]_{B_0} & [f(e_2)]_{B_0} \\ | & | \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}$$

This map takes a vector in $\mathbb{R}^2$ to a vector in $\mathbb{R}^2$.



Put $v_1 = (1, 2)$ and $v_2 = (1, 1)$. Then

$$f(v_1) = (2, 4) = 2v_1$$
$$f(v_2) = (-1, -1) = -v_2$$

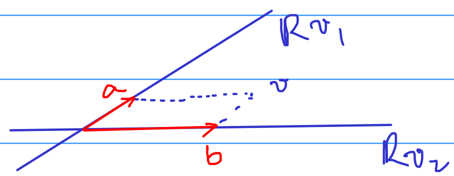


Every vector on the line parallel to v_1 (which we denote by $\mathbb{R}v_1$) is scaled by factor 2. Every vector on the line parallel to v_2 (which we denote by $\mathbb{R}v_2$) is scaled by factor -1.

$$\mathbb{R}^2 = \underbrace{\mathbb{R}v_1}_{\text{eigenspace}} \oplus \underbrace{\mathbb{R}v_2}_{\text{eigenspace}}$$

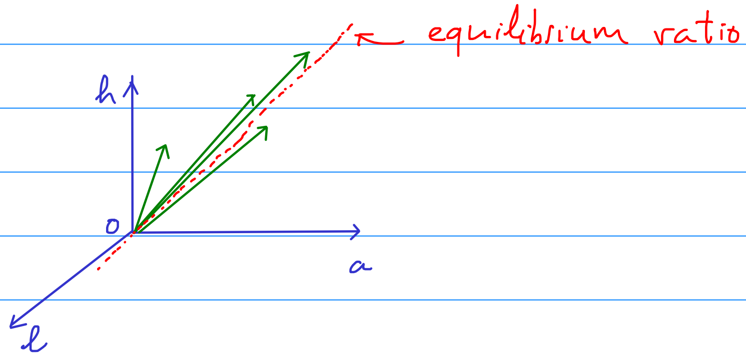
direct sum

A high dimension vector space is split into smaller subspaces so that f behaves more simply on each subspace. This is the overall idea of vector space decomposition.



The circle sum $\oplus$, called direct sum, hasn't been defined yet. But it suggests that each vector $v \in \mathbb{R}^2$ can be expressed uniquely as a sum $v = a + b$ where $a \in \mathbb{R}v_1$ and $b \in \mathbb{R}v_2$. The diagonalization problem is a type of decomposition problem.

The problem of finding eigenvectors is quite natural in many applications. For example, consider a population of lions, antelopes and hyenas living together. At the end of each year, people count the population of each species. Thus, each year corresponds to a triple $(l, a, h) \in \mathbb{R}^3$.



Under a certain rule of population growth (see for example Lab 4 of the class Math 341 in Fall 2018, posted on the website people.oregonstate.edu/~phamt3), the ratio $l : a : h$ converges to a fixed ratio. This is an equilibrium ratio.

Let's make rigorous the idea of "summing" two vector spaces.

Def:

Let V be a vector space, and U and W be subspaces of V. The sum of U and W is defined as the set

$$U + W = \{u + w : u \in U, w \in W\}.$$

In other words, the set $U+W$ is the collection of all vectors in V that can be written as the sum of a vector in U and a vector in W.

Observations:

1) $U+W$ is a subspace of V.

To show this, we first notice that $U+W$ is a subset of V.

We only need to check 3 things:

- $0 \in U+W$:

  This is true because $0 = \underbrace{0}_{\in U} + \underbrace{0}_{W}$

- $U+W$ is closed under addition:

  Let $v_1, v_2 \in U+W$. We want to show $v_1+v_2 \in U+W$.

  Because $v_1 \in U+W$, we can write

  $$v_1 = u_1 + w_1$$

  where $u_1 \in U$ and $w_1 \in W$. Similarly, $v_2 = u_2 + w_2$ where $u_2 \in U$ and $w_2 \in W$. Then

  $$v_1 + v_2 = \underbrace{(u_1+u_2)}_{\in U} + \underbrace{(v_1+v_2)}_{\in W}$$

  Thus, $v_1 + v_2 \in U+W$.

- $U+W$ is closed under scaling:

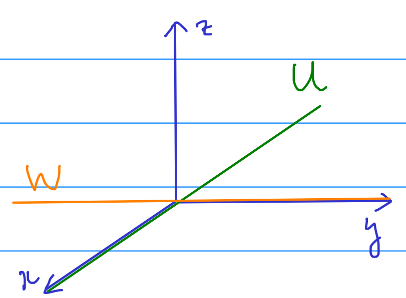
  similar proof.

2) $U \subset U+W$, $W \subset U+W$.

This is true because any vector $u \in U$ can be written as $u = \underbrace{u}_{\in U} + \underbrace{0}_{\in W} \in U+W$.

3) $U+W$ is the smallest subspace of V that contains both U and W. In other words, and subspace of V that contains U and W must contain $U+W$.

Ex:



$V = \mathbb{R}^3$

$U = x$-axis

$W = y$-axis

Then $U+W = xy$-plane.