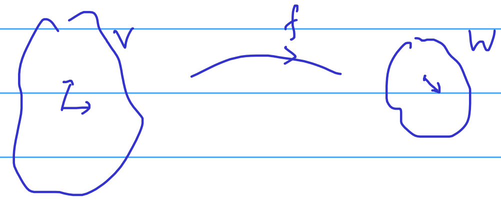
Lecture 15 (10/28/2019)

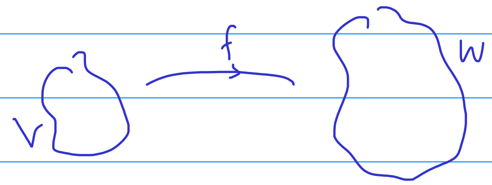
$f: V \to W$ linear

If $\dim V > \dim W$ then f is not monomorphic.



"Large blanket V has to be folded to put in the box W."

If $\dim V < \dim W$ then f is not epimorphic.



"W is too big to be covered by V."

Theorem: If $\dim V = \dim W < \infty$, then monomorphism, epimorphism, isomorphism are equivalent.

Why so?

Suppose $f: V \to W$ is a monomorphism. By rank-nullity theorem,

$$\underbrace{\dim \operatorname{null}(f)}_{\substack{=0 \text{ since} \\ f \text{ is monomorphic}}} + \dim \operatorname{range}(f) = \underbrace{\dim V}_{=\dim W}.$$

Then $\dim \operatorname{range}(f) = \dim W$. Because $\operatorname{range}(f)$ is a subspace of W and has the same dimension as W, it must be equal to W. Hence, f is epimorphic.

Next, we ~~give~~ discuss some quick ways to check if a linear map $f: V \to W$ is monomorphic / epimorphic / isomorphic.

- Check if f is monomorphic:

If $\dim V > \dim W$ then conclude that f is not monomorphic.

Otherwise, one can attempt to show that f is monomorphic by the following methods:

1) Use definition, i.e. show that $null(f) = \{0\}$.
For this method, one can start the proof by saying: "let $v \in V$ such that $f(v) = 0$. We want to show $v = 0$."

2) Check if $\dim range(f) = \dim V < \infty$:
Once this is shown, $null(f) = \{0\}$ because of rank-nullity theorem:

$$\dim null(f) + \underbrace{\dim range(f)}_{\dim V} = \dim V$$

For this method, one can start by writing

$$range(f) = \{\cdots\}$$
$$= span\{\cdots\} \quad \text{(try to find a spanning set)}$$

Then try to find a basis of $range(f)$. It is usually this spanning set (but one needs to check if the spanning set is linearly independent.)
Then one compares the dimension of $range(f)$ with dimension of V.

- Check if f is epimorphic:
  If $\dim V < \dim W$ then conclude that f is not epimorphic
  Otherwise, one can attempt to show that f is epimorphic by the following methods:

1) use definition, i.e. show that $range(f) = W$.
Since $range(f)$ is a subspace of W, it suffices to show that

$$\dim range(f) = \dim W.$$

For this method, one can start by writing

$$range(f) = \{\cdots\}$$
$$= span\{\cdots\} \quad \text{(try to find a spanning set)}$$

Then try to find a basis of $range(f)$. It is usually this spanning set (but one needs to check if the spanning set is linearly independent.)
Then one compares the dimension of $range(f)$ with dimension of W.

2) Check if f is onto.

For this method, one can start by writing: "let $w \in W$. We want to find $v \in V$ such that $f(v) = w$."

- Check if f is isomorphic:

If $\dim V \neq \dim W$ then conclude that f is not isomorphic. Otherwise, one can attempt to show that f is isomorphic by showing that f is monomorphic (or epimorphic). Note that in this case ($\dim V = \dim W$), isomorphism is equivalent to monomorphism and is equiv. to epimorphism.

One can start, for example, by writing that: "Let $v \in V$ such that $f(v) = 0$. We want to show $v = 0$."

See examples on the worksheets.