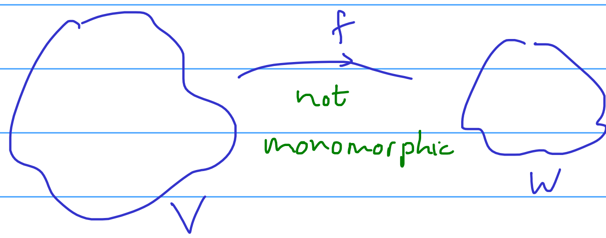
Lecture 14 (10/25/2019)

Last time, we mentioned the following:

If $f: V \to W$ is linear and $\dim V > \dim W$ then f is never one-to-one (i.e. not a monomorphism).



V is too big to fit in W without being "folded".

For example, one cannot find a monomorphism $f: \mathbb{R}^2 \to \mathbb{R}$. A heuristic argument is that $\mathbb{R}$ is too small for f to not repeat itself. $\mathbb{R}$ is too small to contain all linear structure of $\mathbb{R}^2$. Rigorous argument:

$$\dim \operatorname{null}(f) + \underbrace{\dim \operatorname{range}(f)}_{\leq 1} = \underbrace{\dim \mathbb{R}^2}_{2}$$

Then $\dim \operatorname{null}(f) \geq 1$. Then $\operatorname{null}(f)$ is not the trivial vector space $\{0\}$. Therefore, f is not one-to-one (hence, not a monomorphism.)

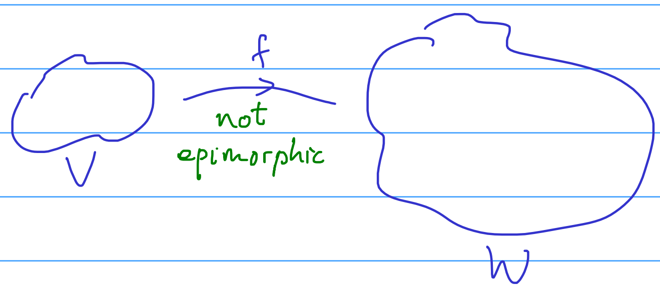
Another way to explain this phenomenon is to use the theorem we stated last time: a monomorphism must preserve linear independence. Let $B = \{v_1, v_2, \dots, v_n\}$ be a basis of V. For f to be monomorphic, the set $\{f(v_1), f(v_2), \dots, f(v_n)\}$ must be linearly ind. in W. If $\dim W < n$ then this cannot be true.

* Question: does there exist a continuous one-to-one map f from $\mathbb{R}^2$ to $\mathbb{R}$?

The answer is no. The proof requires some background on topology, which we don't discuss here.

* General result:

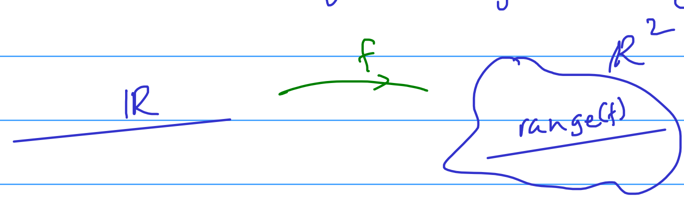
If $f: V \to W$ is linear and $\dim V < \dim W$ then f is not onto (i.e. not an epimorphism).



"W is too big to be covered by V."

Ex: $f: \mathbb{R} \to \mathbb{R}^2$ linear.

Then f is never an epimorphism because $\mathbb{R}^2$ is too large to be covered by $\mathbb{R}$. Rigorous argument:



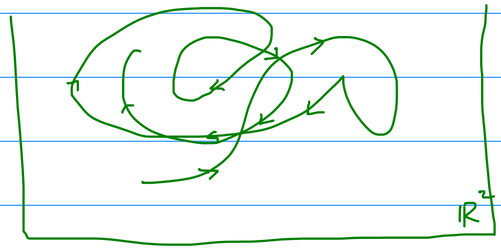
By rank-nullity theorem,

$$\dim \operatorname{null}(f) + \dim \operatorname{range}(f) = \underbrace{\dim \mathbb{R}}_{1}$$

Thus, $\dim \operatorname{range}(f) \leq 1$. In other words, range$(f)$ is at most 1-dimensional. This implies range(f) is strictly smaller than $W = \mathbb{R}^2$. Therefore, f is not onto.

Another way to explain this phenomenon is to use the theorem we stated last time: an epimorphism must preserve spanning set. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. For f to be epimorphic, the set $\{f(v_1), f(v_2), \ldots, f(v_n)\}$ must span W. If $\dim W > n$ then this cannot be true (there are not enough vectors to span W).

* Note that if the condition "f is linear" is relaxed to "f is continuous" then it is possible to find a continuous function $f: \mathbb{R} \to \mathbb{R}^2$ that is onto. This is called a space-filling curve.



See worksheet for some practice.