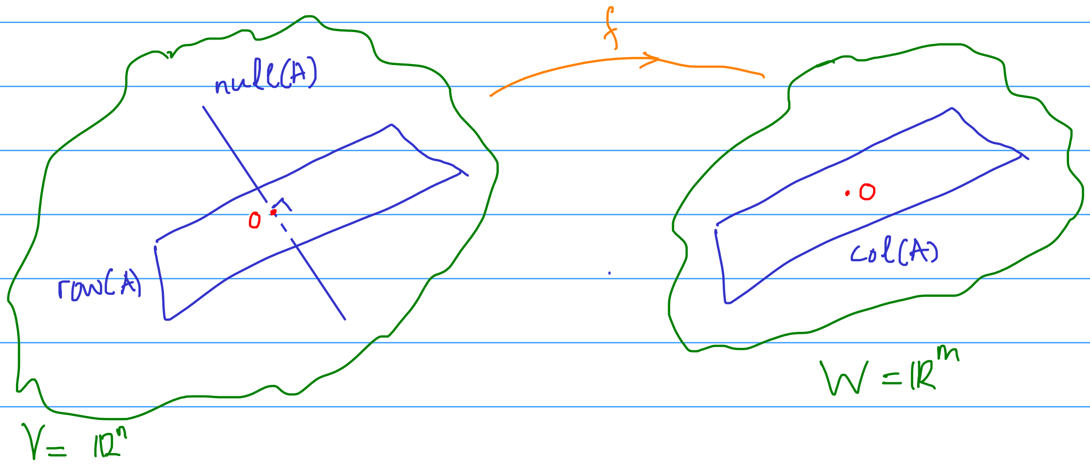
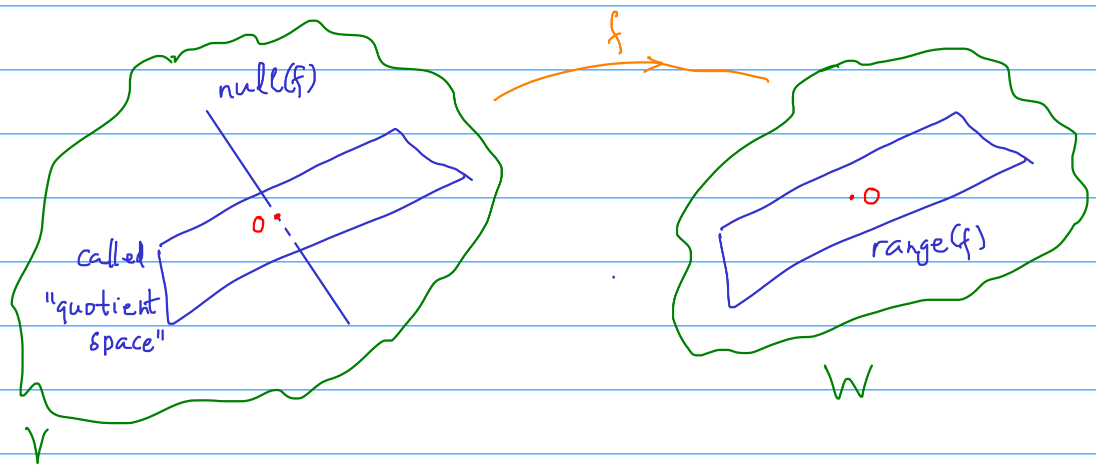
<u>Lecture 12 (10/21/2019)</u>

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map represented by matrix A.



Rank-nullity theorem: nullity + rank = dim V.

For a general linear map $f: V \to W$, the picture is still the same, except that one doesn't have the perpendicularity (since we haven't defined the concept of "angles" for vector space).



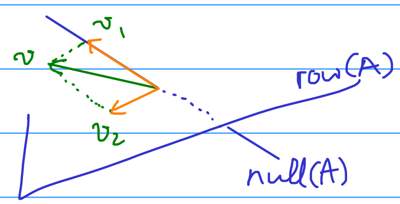
Rank-nullity theorem: nullity + rank = dim V.

The null space of f consists of all vectors in V that get mapped to 0. The "size" (or dimension) of null(f) indicates the 'redundancy' of f.

The range of f consists of all possible values of f. In some sense, the size (or dimension) of range(f) indicates the 'richness' of f. The equality

$$\text{rank} + \text{nullity} = \dim V \qquad (*)$$

suggests that the more redundant f is (i.e. the larger the nullity), the poorer information it can provide (i.e. the smaller the rank). Thus, $(*)$ can be seen as "conservation of dimension".



From another perspective, $(*)$ suggests that each vector $v \in V$ can be split into two parts: $v = v_1 + v_2$, where $v_1 \in \text{null}(f)$ and v_2 is in the "quotient space". The first part gets mapped to 0. The second part is never mapped to 0 unless $v = 0$.

The following example is an application of rank-nullity theorem.

Ex:

Show that for each polynomial p of degree ≤ 3, there exists a polynomial q of degree ≤ 4 such that $5q'' + 3q' = p$.

Let V be the vector space of all poly. of degree ≤ 4,
W " " " " " ≤ 3.

Define a map $F: V \to W$ by $F(u) = 5u'' + 3u'$.
This is a well-defined map because for each $u \in V$, the function $5u'' + 3u'$ is a polynomial of degree ≤ 3 (thus belonging to W).

Moreover, F is a linear map. One can check this fact by checking if F is additive and scalar multiplicative.

Let us find the null space of F.

$$\text{null}(F) = \{u \in V : F(u) = 0\}$$
$$= \{u \in V : 5u'' + 3u' = 0\}$$

Consider a polynomial u satisfying $5u'' + 3u' = 0$. Integrating both sides:

$$5u' + 3u = C. \qquad (**)$$

We claim that u must be a constant. Indeed, suppose by contradiction that u is of degree $n \geq 1$. Then $3u$ is of degree n, and $5u'$ of degree $n-1$. Thus, LHS of $(**)$ is of degree $n \geq 1$, while RHS of $(**)$ is of degree 0. This is a contradiction. Therefore, any polynomial satisfying $(**)$ must be a constant. We get

$$\text{null}(F) = \{u \in V : u \text{ is a constant}\}$$
$$= \text{span}\{1\}$$

$\text{Null}(F)$ is a 1-dimensional vector space with basis $B = \{1\}$.

By rank-nullity theorem,

$$\underbrace{\dim \text{null}(F)}_{1} + \dim \text{range}(F) = \underbrace{\dim V}_{5}$$

Thus, $\dim \text{range}(F) = 4$. Since $\text{range}(F)$ is a 4-dimensional subspace of W, which is also 4-dimensional, we have

$$\text{range}(F) = W.$$

In other word, and polynomial $p \in W$ must lie in $\text{range}(F)$. This means that there is a polynomial $q \in V$ such that

$$p = F(q) = 5q'' + 3q'.$$