Lecture 10 (10/16/2019)

$f: V \longrightarrow W$ linear map.

V ... basis $B_1 = \{v_1, v_2, \ldots, v_n\}$

W .... basis $B_2 = \{w_1, w_2, \ldots, w_m\}$

The matrix representation of f in these bases is

$$[f]_{B_2,B_1} = \begin{bmatrix} | & & | \\ [f(v_1)]_{B_2} & \cdots & [f(v_n)]_{B_2} \\ | & & | \end{bmatrix}$$

Important property: for any $v \in B_1$,

$$\underbrace{[f(v)]_{B_2}}_{m\times 1} = \underbrace{[f]_{B_2,B_1}}_{m\times n} \underbrace{[v]_{B_1}}_{n\times 1}$$

With this property, one can work on linear maps on general vector spaces through coordinate vectors.

Operations on linear maps

Let f and g be linear maps from V to W (over a field F).
Then $f, g \in W^V$ (the vector space of all functions from V to W).
We already know how to add f and g and how to scale each map:

$$(f+g)(v) = f(v) + g(v) \qquad \forall v \in V$$

$$(cf)(v) = cf(v) \qquad \forall v \in V, \ \forall c \in F$$

In coordinates:

$$[f+g]_{B_2,B_1} = [f]_{B_2,B_1} + [g]_{B_2,B_1} \quad \text{(addion of matrices)}$$

$$[cf]_{B_2,B_1} = c\,[f]_{B_2,B_1} \quad \text{(scaling a matrix)}$$

What is the operation of linear maps that corresponds to matrix multiplication? Answer: map composition.

Consider two linear maps:

$$V_0 \xrightarrow{f} V_1 \xrightarrow{g} V_2$$

Let B_0, B_1, B_2 be bases of V_0, V_1, V_2 respectively. It is good exercise to show that the composite map $g \circ f$ is also a linear map. Let us assume this property.

Because $g \circ f$ is a linear map from V_0 to V_2, the matrix representation of $g \circ f$ in basis B_0 and B_2 is $[g \circ f]_{B_2, B_0}$.

We have an interesting property as follows:

$$[g \circ f]_{B_2, B_0} = \underbrace{[g]_{B_2, B_1} [f]_{B_1, B_0}}_{\text{matrix multiplication}}$$

Note that the rule of multiplication of matrices was defined such that matrix multiplication is compatible with composition of linear maps. We observe that operations on linear maps are compatible with operations on their representation matrices.

In Math 341, we learned null space and range space of a matrix: if A is an $m \times n$ matrix then

$$\text{null}(A) = \{ v \in \mathbb{R}^n : Av = 0 \}$$

$$\text{range}(A) = \{ Av : v \in \mathbb{R}^n \}$$

In some textbooks, $\text{null}(A)$ is denoted as $\ker(A)$ (kernel), $\text{range}(A)$ is denoted as $\text{im}(A)$ (image).

The corresponding definitions in linear maps are:

$f: V \to W$ linear

$$\text{null}(f) = \{ v \in V : f(v) = 0 \}$$

$$\text{range}(f) = \{ f(v) : v \in V \}$$

We will consider some examples next time.