Lecture 1 (9/25/2019)

Math 341 .... Linear alg. .... Math 342

theme

$\mathbb{R}^n$ ...... Vector spaces ..... $\mathbb{R}^n$ and more general spaces

matrices ...... linear maps .... more general than matrices

Math 342 is an expansion and generalization of Math 341. If one does something to a matrix, he is doing something to a linear map. (Matrix represents a linear map and vice versa).

Linear maps: $f: \mathbb{R}^n \to \mathbb{R}^m$

$$\begin{cases} f(x+y) = f(x) + f(y) \\ f(cx) = cf(x) \end{cases}$$

A linear map can be represented by a matrix of size $m \times n$.

$$f(x) = \underset{m\times n}{A}\ \underset{n\times 1}{x}$$

- Solving system of linear equations .... $Ax = b$

  $\longrightarrow f(x) = b$

- Addition ... $A+B$

  f is represented by A, g by B

  $A+B$ is represented by $f+g$

- Scaling (scalar multiplication) ..... cA. This matrix represents the map cf.

- Multiplication .... AB, representing the map $f \circ g$ (function composition)

- Inverse .... A^{-1} represents the inverse map of f.

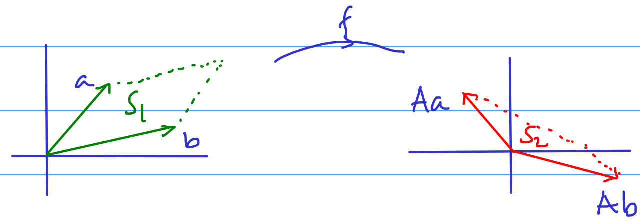
  Recall: f and g are inverses of each other if

  $x \xrightarrow{f} y \xrightarrow{g} x$ (or $g \circ f = id$)

  $y \xrightarrow{g} x \xrightarrow{f} y$ (or $f \circ g = id$)

If f is represented by A, g by B then
$f \circ g$ is represented by AB
$g \circ f$ is represented by BA
id is represented by I_n
g is the inverse map of f if and only if $AB = BA = I_n$.
In other words, B is equal to A^{-1}.

- Eigenvectors .... $Ax = \lambda x$
  $\rightarrow f(x) = \lambda x$
  x is an invariant direction under f.
- Determinant

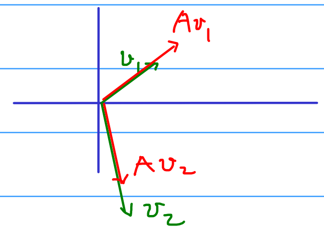


$$\det A = \pm \frac{S_2}{S_1}$$

Plus sign is taken if the rotation from Aa to Ab is the same as the rotation from a to b.
Otherwise, the minus sign is taken.
From the picture, we choose the plus sign.
$\rightarrow$ Determinant indicates how much a linear map stretches the area.

- Diagonalization:
  Matrix A of size $n \times n$ is diagonalizable if $\mathbb{R}^n$ has a basis consisting of eigenvectors of A.
  For example: let A be a diagonalizable matrix of size 2×2.
  Then $\mathbb{R}^2$ has a basis $\{v_1, v_2\}$ such that

Av_1 is parallel to v_1,
Av_2 is parallel to v_2.



Consequently, each vector $v \in \mathbb{R}^2$ has a decomposition

$$v = c_1 v_1 + c_2 v_2$$

This decomposition is invariant under A in sense that

$$Av = c_1 Av_1 + c_2 Av_2$$
$$= (c_1\lambda_1) v_1 + (c_2\lambda_2) v_2$$

We see that Av is naturally decomposed into a v_1-component and a v_2-component.

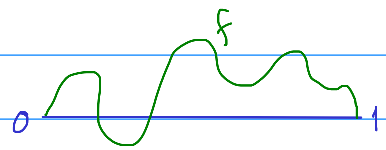
If one pick a different basis of $\mathbb{R}^2$, say $\{e_1, e_2\}$, then the decomposition is no longer invariant under A:

$$v = d_1 e_1 + d_2 e_2$$
$$Av = d_1 \underbrace{Ae_1}_{\substack{\text{not} \\ \text{parallel to } e_1}} + d \underbrace{Ae_2}_{\cdots}$$

Let us consider an example where the idea (of decomposition) in linear algebra is helpful.

Consider the set $S = \{ f : [0,1] \to \mathbb{R},\ f(0) = f(1) = 0,\ f \text{ is smooth} \}$.

A member of S is a function! One can think of a function $f \in S$ as a string fixed at two ends.



The set S is closed under addition: if one adds two members in S, one gets another member in S.
Also, S is closed under scaling: if one scale a member of S by a constant factor, one gets another member in S.

This makes S look like $\mathbb{R}^n$. Let us apply the eigenvalue eigenvector to S as if S were $\mathbb{R}^n$.

* Consider the map $T(f) = f''$.
Eigenvalue, eigenvector of T satisfy $T(f) = \lambda f$

$\rightsquigarrow$ solve the differential eq. $\begin{cases} f'' - \lambda f = 0 \\ f(0) = f(1) = 0 \end{cases}$

we don't want the solution $f \equiv 0$ because eigenvectors must be nonzero.
The only case where the diff. eq. gives a nontrivial sol. is when $\lambda = -n^2\pi^2$, where n is positive integer.
In this case, $f_n(x) = \sin(n\pi x)$.

Assume that T is a diagonalizable linear map. Then $\{f_1, f_2, f_3, \ldots\}$ would constitutes a basis of S. (In this case, S is infinite dimensional.)
Each $f \in S$ has a decomposition $f(x) = \sum_{k=1}^{\infty} c_k f_k(x)$

$$= \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

This is known as Fourier series - an important tool in signal processing. It decomposes an quite arbitrary signal f into

simple "modes", i.e. signals of sine form.
In the course, we will use the techniques in Math 341 for more general settings. We try to learn the most out of our available tools.

* Strictly speaking, the def. of S should include $f^{(2k)}(0) = f^{(2k)}(1) = 0$ for all $k = 0, 1, 2, \ldots$ so that T takes a member of S to a member of S.