Homework 7

Answer Key

1. Let $T\colon V \to V$ be a linear map. Show that null(T) and range(T) are invariant subspaces under T.

   Solution:

   Let v be an arbitrary element of null(T). Then $T(u) = 0$ and $0 \in$ null(T), since $T(0) = 0$. Therefore $T(u) \in$ null(T) so null(T) is an invariant subspace under T.

   Naturally for any element $u \in V$, the element $T(u)$ is contained in range(T). Now let v be an arbitrary element of range$(T) \subset V$. Therefore $T(v) \in$ range(T), so range(T) is an invariant subspace under T.

2. Let $T\colon V \to V$ be a linear map. For each $n \geq 1$, denote by T^n the composition mmapping $T \circ T \circ \cdots \circ T$ (n times). Show that null$(T) \subset$ null(T^2).

   Solution: Let $v \in$ null(T). Then

   $$T^2(v) = T(T(v)) = T(0) = 0$$

   so $v \in$ null(T^2). Therefore null$(T) \subset$ null(T^2).

3. Let

   $$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

   Put

   $$\begin{aligned} V_1 &= \{v : Av = (-1)v\} \\ V_2 &= \{v : Av = 0v\} \\ V_3 &= \{v : Av = 2v\} \end{aligned}$$

   Show that $V_1 \oplus V_2 \oplus V_3 = \mathbb{R}^4$.

   Solution: This is the problem of finding eigenvectors. First conisder V_1. The elements of V_1 are the vectors $v \in \mathbb{R}^4$ that satisfy the equation

   $$(A - (-1)I)v = 0 \qquad \to \qquad (A + I)v = 0$$

   where I is the 4×4 identity matrix. This becomes

   $$\left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

   $$\downarrow$$

   $$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Write this equation in augmented form and row-reduce:

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 & 0 \\ 6 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{swap } R_1 \text{ and } R_3} \left[\begin{array}{cccc|c} 6 & 2 & 1 & 0 & 0 \\ 6 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cccc|c} 6 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{cccc|c} 6 & 0 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3} R_1 \to R_1} \left[\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This augmented system gives the equations

$$\begin{aligned} 2v_1 + v_3 &= 0 \\ v_2 - v_3 &= 0 \end{aligned}$$

We can then write $v_2 = v_3 = -2v_1$, so the elements of V_1 look like

$$v = \begin{bmatrix} v_1 \\ -2v_1 \\ -2v_1 \\ v_4 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -2 \\ -2 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $v_1, v_4 \in \mathbb{R}$. Therefore

$$B_1 = \{(1, -2, -2, 0), (0, 0, 0, 1)\}$$

is a basis for V_1.

The process is similar for V_2 and V_3. I will omit some of the details.

$v \in V_2$ satisfies $(A - 0I)v = 0$. That is, $Av = 0$.

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow{\text{row-reduce}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so $v \in V_2$ looks like

$$v = \begin{bmatrix} 0 \\ 0 \\ v_3 \\ 0 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and a basis for V_2 is

$$B_2 = \{(0, 0, 1, 0)\}.$$

$v \in V_3$ satisfies $(A - 2I)v = 0$.

$$\left[\begin{array}{cccc|c} -3 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 \\ 6 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{\text{row-reduce}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so $v \in V_3$ looks like

$$v = \begin{bmatrix} 0 \\ v_2 \\ v_2 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

and a basis for V_3 is

$$B_3 = \{(0, 1, 1, 0)\}.$$

Now conisder the set

$$B = B_1 \cup B_2 \cup B_3 = \{(1, -2, -2, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 1, 0)\}$$

Let the elements of B be the rows of a matrix:

$$\begin{bmatrix} 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

row reduction gives the identity matrix, so the rows are linearly independen. Therefore B is linearly independent, so it is a basis for $V_1 + V_2 + V_3$. Then

$$\dim(V_1 + V_2 + V_3) = 4 = 2 + 1 + 1 = \dim(V_1) + \dim(V_2) + \dim(V_3)$$

so the sum is a direct sum. Since $\dim(V_1 \oplus V_2 \oplus V_3) = 4$, we must have that $V_1 \oplus V_2 \oplus V_3 = \mathbb{R}^4$.

4. Consider a subspace of $P_3(\mathbb{R})$

$$V_1 = \{u \in P_2(\mathbb{R}) : u(1) = u'(1) = 0)\}.$$

Find a subspace V_2 of P_3 such that $V_1 \oplus V_3 = P_3$.

Solution: Write a polynomial u as

$$u(x) = a + bx + cx^2 + dx^3.$$

The condition $u(1) = u'(1) = 0$ gives

$$a + b + c + d = 0 \qquad \text{and} \qquad b + 2c + 3d = 0.$$

Write this system in augmented form and row-reduce:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

This gives the equations

$$a = c + 2d \qquad \text{and} \qquad b = -2c - 3d.$$

It is helpful to use the basis $\{1, x, x^2, x^3\}$ for P_3 to rewrite u as

$$(a, b, c, d).$$

Then

$$\begin{aligned} V_1 &= \{(a, b, c, d) \in \mathbb{R}^4 : a = c + 2d,\ b = -2c - 3d\} \\ &= \{(c + 2d, -2c - 3d, c, d) : d, a_4 \in, \mathbb{R}\} \\ &= \{c(1, -2, 1, 0) + d(2, -3, 0, 1) : c, d \in \mathbb{R}\} \end{aligned}$$

so

$$B_1 = \{(1, -2, 1, 0), (2, -3, 0, 1)\}$$

is a basis for V_1. Let the elements of B be the rows of a matrix and row-reduce:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{\text{row-reduce}} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

The rows of the row-reduced matrix

$$B_1 = \{(1, 0, -3, 2), (0, 1, -2, 1)\}$$

are also a basis for V_1. To find a basis for V_2, find the non-pivot columns of the row-reduced matrix. These are columns 3 and 4, so let

$$B_2 = \{e_3, e_4\} = \{(0, 0, 1, 0), (0, 0, 0, 1)\}$$

be a basis for V_2. Rewriting as polynomials, the basis for V_2 is $\{x^2, x^3\}$.

Let $B = B_1 \cup B_2$. Writing the elements of B as rows of a matrix gives

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Row-reduction gives the identity matrix, so the rows are linearly independent. Therefore

$$\dim(V_1 + V_2) = 4 = 2 + 2 = \dim(V_1) + \dim(V_2)$$

so the sum is a direct sum. Since $\dim(V_1 \oplus V_2) = 4 = \dim(P_3)$ we must have that $V_1 \oplus V_2 = P_3$.

5. Consider two subspaces of $\mathbb{C}^3$:

$$V_1 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1(1+i) + 2z_2 = 0\}$$
$$V_2 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_2 + (2-i)z_3 = 0\}.$$

Find a basis of $V_1 + V_2$. Is it a direct sum?

Solution: First find bases for V_1 and V_2:

$$\begin{aligned} V_1 &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1(1+i) + 2z_2 = 0\} \\ &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_2 = -\tfrac{1+i}{2} z_1\} \\ &= \{(z_1, -\tfrac{1+i}{2} z_1, z_3) : z_1, z_3 \in \mathbb{C}\} \\ &= \{z_1(1, -\tfrac{1+i}{2}, 0) + z_3(0, 0, 1) : z_1, z_3 \in \mathbb{C}\} \end{aligned}$$

so

$$B_1 = \{(1, -\tfrac{1+i}{2}, 0), (0, 0, 1)\}$$

is a basis for V_1.

$$\begin{aligned} V_2 &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_2 + (2-i)z_3 = 0\} \\ &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_2 = (-2+i)z_3\} \\ &= \{(z_1, (-2+i)z_3, z_3) : z_1, z_3 \in \mathbb{C}\} \\ &= \{z_1(1, 0, 0) + z_3(0, -2+i, 1) : z_1, z_3 \in \mathbb{C}\} \end{aligned}$$

so

$$B_2 = \{(1, 0, 0), (0, -2+i, 1)\}$$

is a basis for V_2.

Now let $S = B_1 \cup B_2$, so the elements of S span $V_1 + V_2$. Write the elements of S as the rows of a matrix and row-reduce:

$$\begin{bmatrix} 1 & -\frac{1+i}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -2+i & 1 \end{bmatrix} \xrightarrow{\text{reorder rows}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -\frac{1+i}{2} & 0 \\ 0 & 0 & 1 \\ 0 & -2+i & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1+i}{2} & 0 \\ 0 & 0 & 1 \\ 0 & -2+i & 1 \end{bmatrix}$$

$$\xrightarrow{-(1-i)R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2+i & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 + (2-i)R_2 \to R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_3 \to R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The rows of the reduced matrix also span $V_1 + V_2$. Therefore

$$B = \{(1,0,0), (0,1,0), (0,0,1)\}$$

is a linearly independent set spanning $V_1 + V_2$, so B is a basis for $V_1 + V_2$.

Notice that $\dim(V_1) = \dim(V_2) = 2$, but $V_1 + V_2 = \mathbb{C}^3$ so $\dim(V_1 + V_2) = 3$. Therefore

$$\dim(V_1 + V_2) \neq \dim(V_1) + \dim(V_2),$$

so $V_1 + V_2$ is not a direct sum of V_1 and V_2.