# Homework 6

**Answer Key**

1. Let $V$ be a vector space over a field of numbers $F$ (which could be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$). Let $U$ be a subspace of $V$. Show that $U + U = U$. Under what condition of $U$ is this sum a direct sum?

   **Solution:** To show $U + U = U$ we want to show that $U + U \subseteq U$ and $U \subseteq U + U$.

   (i) Let $u + v$ be an arbitrary element for $U + U$ where $u, v \in U$. Then $u + v \in U$, since $U$ is closed under addition (because $U$ is a vector space). Therefore $U + U \subseteq U$.

   (ii) Let $u$ be an arbitrary element of $U$. Then $u = u + 0 \in U + U$ since $0 \in U$ (again, because $U$ is a vector space). Therefore $U \subseteq U + U$.

   The sum is a direct sum if the intersection of the summands is the trivial vector space $\{0\}$. In our case we would need $U \cap U = \{0\}$. Since $U \cap U = U$, the only way for this to be a direct sum is if $U = \{0\}$.

2. Let $U = \{(x, y, y, x) : x, y \in \mathbb{R}\}$. This is a subspace of $\mathbb{R}^4$. Find a subspace $V$ of $\mathbb{R}^4$ such that $U \oplus V = \mathbb{R}^4$.

   **Solution:** First, find a basis for $U$.

$$U = \{x(1, 0, 0, 1) + y(0, 1, 1, 0) : x, y \in \mathbb{R}\}$$

   so a basis for $U$ is $\{(1, 0, 0, 1), (0, 1, 1, 0)\}$. Now form the matrix $A$ using the basis vectors as rows:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

   Next we would compute RREF($A$), but notice that $A$ is already in reduced row echelon form (RREF($A$) $= A$). Find the non-pivot columns of RREF($A$):

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$
$$\qquad\qquad\uparrow\ \ \uparrow$$

   These are columns 3 and 4. Let $V$ be the span of standard basis vectors with a 1 in the coordinates corresponding to the non-pivot columns:

$$V = \operatorname{span}(\{e_3, e_4\}) = \operatorname{span}(\{(0, 0, 1, 0), (0, 0, 0, 1)\}) = \{(0, 0, a, b) : a, b \in \mathbb{R}\}$$

   To see that $U + V = \mathbb{R}^4$, notice that

$$e_1 = (1, 0, 0, 1) - (0, 0, 0, 1) \in U + V$$
$$e_2 = (0, 1, 1, 0) - (0, 0, 1, 0) \in U + V$$

   So $e_1, e_2 \in U + V$. Since $e_3, e_4 \in V$, they are also in $U + V$. Therefore $U + V$ contains the standard basis $\{e_1, e_2, e_3, e_4\}$ for $\mathbb{R}^4$, so $\mathbb{R}^4 \subseteq U + V$. Since $U, V \subseteq \mathbb{R}^4$ we conclude that $U + V \subseteq \mathbb{R}^4$ and thus $U + V = \mathbb{R}^4$.

   Remember that a sum of finite-dimensional vector spaces $U$ and $V$ is a direct sum if

$$\dim(U + V) = \dim(U) + \dim(V).$$

   Since $U$ and $V$ have dimension 2 and $U + V = \mathbb{R}^4$ has dimension 4, we see that this equality holds, so $U + V$ is a direct sum of $U$ and $V$.

3. Consider two vector spaces

$$U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ a + d = b + c = 0 \right\},$$
$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ a = d = 0 \right\}.$$

(a) Find a basis for $U + V$.

**Solution:** It is helpful to write these matrices as vectors:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a, b, c, d)$$

We first need to find bases for $U$ and $V$.

$$\begin{aligned} U &= \{(a, b, c, d) : a, b, c, d \in \mathbb{R},\ a + d = b + c = 0\} \\ &= \{(a, b, c, d) : a, b, c, d \in \mathbb{R},\ d = -a,\ c = -b\} \\ &= \{(a, b, -b, -a) : a, b \in \mathbb{R}\} \\ &= \{a(1, 0, 0, -1) + b(0, 1, -1, 0) : a, b \in \mathbb{R}\} \end{aligned}$$

so a basis for $U$ is $B_U = \{(1, 0, 0, -1), (0, 1, -1, 0)\}$.

$$\begin{aligned} V &= \{(a, b, c, d) : a, b, c, d \in \mathbb{R},\ a = d = 0\} \\ &= \{(0, b, c, 0) : b, c \in \mathbb{R}\} \\ &= \{b(0, 1, 0, 0) + c(0, 0, 1, 0) : b, c \in \mathbb{R}\} \end{aligned}$$

so a basis for $V$ is $B_V = \{(0, 1, 0, 0), (0, 0, 1, 0)\}$.

The union of the two bases $B_U \cup B_V$ spans the space $U + V$ but it is not necessarily a basis. To find a basis for $U + V$, let the vectors in $B_U \cup B_V$ be the rows of a matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Now find the reduced row echelon form of $A$. You can do this with MATLAB. Here is a reduction by hand:

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} &\xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &\xrightarrow{R_4 - R_3 \to R_4} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\xrightarrow{R_2 + R_3 \to R_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A) \end{aligned}$$

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The non-zero rows of RREF($A$) form a basis for $U+V$:

$$B_{U+V} = \{(1,0,0,-1),(0,1,0,0),(0,0,1,0)\},$$

however, we should rewrite these as matrices (since that is how $U$ and $V$ were written):

$$B_{U+V} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

(b) Is $U+V$ a direct sum?

**Solution:** No. $\dim(U) = 2$ and $\dim(V) = 2$, but

$$\dim(U+V) = 3 \neq \dim(U) + \dim(V).$$

Therefore $U+V$ is not a direct sum of $U$ and $V$.

4. Let $V = \{z \in \mathbb{C} : z(1+i) + 2\overline{z} = 0\}$. Is $V$ a vector space over $\mathbb{C}$?

**Solution:** Remember that any element $z \in \mathbb{C}$ can be written as

$$z = x + iy, \qquad x, y \in \mathbb{R}.$$

The complex conjugate of $z = x + iy$ is

$$\overline{z} = x - iy.$$

We can use this definition to rewrite $V$:

$$\begin{aligned} V &= \{z \in \mathbb{C} : z(1+i) + 2\overline{z} = 0\} \\ &= \{x + iy \in \mathbb{C} : (x+iy)(1+i) + 2(x - iy) = 0,\ x, y \in \mathbb{R}\} \\ &= \{x + iy \in \mathbb{C} : x + iy + ix - y + 2x - 2iy = 0,\ x, y \in \mathbb{R}\} \\ &= \{x + iy \in \mathbb{C} : (3x - y) + i(x - y) = 0,\ x, y \in \mathbb{R}\} \end{aligned}$$

Since $x$ and $y$ are real numbers, the equation

$$(3x - y) + i(x - y) = 0$$

is only satisfied if $3x - y = 0$ and $x - y = 0$. Subtracting the second equation from the first we get $2x = 0$, and hence $x = 0$ and $y = 0$. Therefore

$$V = \{0\},$$

so $V$ is a vector space over $\mathbb{C}$.