Homework 5
Answer Key

Consider the following subspaces of $\mathbb{R}^3$:

$$\begin{aligned} U &= \{(x_1, x_2, x_3) : x_1 = x_2 + x_3\} \\ V &= \{(x_1, x_2, x_3) : x_1 = x_2\} \\ W &= \{(x_1, x_2, x_3) : x_1 = x_2 = x_3\} \end{aligned}$$

1. Find a basis of the intersection $U \cap V$. What is the dimension?

   Solution: We can write the intersection as a subset of $\mathbb{R}^3$ by including the conditions for both U and V:

$$\begin{aligned} U \cap V &= \{(x_1, x_2, x_3) : x_1 = x_2 + x_3,\ x_1 = x_2\} \\ &= \{(x_1, x_1, x_3) : x_1 = x_1 + x_3\} \\ &= \{(x_1, x_1, 0) : x_1 \in \mathbb{R}\} \\ &= \{x_1(1, 1, 0) : x_1 \in \mathbb{R}\} \end{aligned}$$

   The set $B = \{(1, 1, 0)\}$ spans $U \cap V$, and since it is a set of only one non-zero element, B is linearly independent. Therefore B is a basis for $U \cap V$ and the dimension is

$$\dim(U \cap V) = 1.$$

2. Find a basis of $U \cap W$. What is the dimension?

   Solution: We can write the intersection as a subset of $\mathbb{R}^3$ by including the conditions for both U and W:

$$\begin{aligned} U \cap W &= \{(x_1, x_2, x_3) : x_1 = x_2 + x_3,\ x_1 = x_2 = x_3\} \\ &= \{(x_1, x_1, x_1) : x_1 = x_1 + x_1\} \\ &= \{(0, 0, 0)\} \end{aligned}$$

   Therefore $U \cap W$ is the vector space of only the zero element, so a basis for $U \cap W$ is the empty set $B = \varnothing$. The dimension of $U \cap W$ is the size of the empty set:

$$\dim(U \cap V) = 0.$$

3. Show that $U + W = \mathbb{R}^3$.

Solution: We want to show that $U + W$ contains a basis for $\mathbb{R}^3$. We will start by finding bases for U and W separately.

$$\begin{aligned} U &= \{(x_1, x_2, x_3) : x_1 = x_2 + x_3\} \\ &= \{(x_2 + x_3, x_2, x_3) : x_2, x_3 \in \mathbb{R}\} \\ &= \{x_2(1, 1, 0) + x_3(1, 0, 1) : x_2, x_3 \in \mathbb{R}\} \end{aligned}$$

and a basis for U is $\{(1, 1, 0), (1, 0, 1)\}$.

$$\begin{aligned} W &= \{(x_1, x_2, x_3) : x_1 = x_2 = x_3\} \\ &= \{x_1(1, 1, 1) : x_1 \in \mathbb{R}\} \end{aligned}$$

and a basis for W is $\{(1, 1, 1)\}$. The union of these two bases is

$$B = \{(1, 1, 0), (1, 0, 1), (1, 1, 1)\}.$$

Since B is the union of a basis for U and a basis for W, that means $\text{span}(B) = U + W$. We want to check that B is linearly independent (and hence a basis for $\mathbb{R}^3$). Consider the equation

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Writing this as an augmented matrix and using row reduction gives

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] &\xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \\ &\xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \\ &\xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \end{aligned}$$

Therefore $c_1 = c_2 = c_3 = 0$, so B is linearly independent. Since $\mathbb{R}^3$ has dimension 3 and B is a linearly independent subset of $\mathbb{R}^3$ with 3 elements, B must be a basis for $\mathbb{R}^3$. Therefore

$$\mathbb{R}^3 = \text{span}(B) = U + W.$$

4. Show that $V + W = V$.

Solution: The only way for this to be true is if W is a subset of V. Notice that any element $(x_1, x_2, x_3) \in \mathbb{R}^3$ in W satisfies $x_1 = x_2$. This is the only condition required for V, so any element of W is also an element of V.

If $v \in V$ and $w \in W$, then $v + w \in V$ since $w \in W \subseteq V$. Therefore

$$V + W = \{v + w : v \in V,\ w \in W\} \subseteq V.$$

You saw in class that V is always a subset of $V + W$ (just let $w = 0$). Therefore $V + W = V$.