Homework 2

Answer Key

1. Let V be a vector space over a field $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Suppose $v_1, v_2 \in V$ are linearly independent of each other. Show that vectors $v_1 + 2v_2$ and $2v_1 + 3v_2$ are linearly independent.

   Solution: Let $c_1, c_2 \in \mathbb{R}$ and consider the equation

   $$c_1(v_1 + 2v_2) + c_2(2v_1 + 3v_2) = \mathbf{0}$$

   To show these vectors are linearly independent, we must show that $c_1 = 0$ and $c_2 = 0$.

   $$\begin{gathered} c_1(v_1 + 2v_2) + c_2(2v_1 + 3v_2) = \mathbf{0} \\ \downarrow \\ c_1v_1 + 2c_1v_2 + 2c_2v_1 + 3c_2v_2 = \mathbf{0} \\ \downarrow \\ (c_1 + 2c_2)v_1 + (2c_1 + 3c_2)v_2 = \mathbf{0} \end{gathered}$$

   Now since v_1 and v_2 are linearly independent, we must have both the v_1 and v_2 coefficients from this last equation be equal to 0:

   $$\begin{aligned} c_1 + 2c_2 &= 0 \\ 2c_1 + 3c_2 &= 0 \end{aligned}$$

   Now solve the system of equations (there are many ways to do this). Subtract 2 times the first equation from the second equation to get

   $$-c_2 = 0 \qquad \rightarrow \qquad c_2 = 0.$$

   Now that we know $c_2 = 0$, the first equation becomes

   $$c_1 + 0 = 0 \qquad \rightarrow \qquad c_1 = 0.$$

   Therefore $v_1 + 2v_2$ and $2v_1 + 3v_2$ are linearly independent.

2. Show that the functions $y_1 = \sin(x)$, $y_2 = \cos(x)$ and $y_3 = \sin(2x)$ are linearly independent over $\mathbb{R}$.

   Solution: Let $c_1, c_2, c_3 \in \mathbb{R}$ and set an arbitrary linear combination equal to $\mathbf{0}$:

   $$c_1 \sin(x) + c_2 \cos(x) + c_3 \sin(2x) = 0.$$

   This must hold **for all** $\boldsymbol{x} \in \mathbb{R}$. We want to show that $c_1 = c_2 = c_3 = 0$. To find these coefficients, we can plug in different values for x:

   $$\begin{array}{ccccc} x = \frac{\pi}{2} & \rightarrow & c_1 + 0 + 0 = 0 & \rightarrow & c_1 = 0 \\ x = 0 & \rightarrow & 0 + c_2 + 0 = 0 & \rightarrow & c_2 = 0 \\ x = \frac{\pi}{4} & \rightarrow & \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 + c_3 = 0 & \rightarrow & c_3 = 0 \end{array}$$

   (notice that the last equation gives $c_3 = 0$, because we already knew that $c_1 = c_2 = 0$ from the first two equations). Therefore $\sin(x)$, $\cos(x)$, and $\sin(2x)$ are linearly independent.

3. Consider the set

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, a + d = 0 \right\}$$

(a) Show that V is a vector space over $\mathbb{R}$.

Solution: Notice that V is a subset of the vector space $M_{2\times 2}(\mathbb{R})$ of 2×2 real matrices. To show that V is a vector space, we only need to show that it is a subspace of $M_{2\times 2}(\mathbb{R})$. There are three axioms to check:

A0 - Closed under addition: Let $A, B \in V$ and write

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$
$$B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

and

$$\begin{aligned} a_1 + a_2 + d_1 + d_2 &= (a_1 + d_1) + (a_2 + d_2) \\ &= 0 + 0 \qquad \text{since } A, B \in V \\ &= 0 \end{aligned}$$

so $A + B \in V$.

S0 - Closed under scalar multiplication: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$$

and let $\beta \in \mathbb{R}$. Then

$$\beta A = \begin{bmatrix} \beta a & \beta b \\ \beta c & \beta d \end{bmatrix}$$

and

$$\beta a + \beta d = \beta(a + d) = \beta(0) = 0,$$

so $\beta A \in V$.

A3 - V contains the zero vector: The zero vector is the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with every coordinate equal to 0. Then $a + d = 0 + 0 = 0$, so the zero vector is in V.

(b) Find a basis for V.

Solution: A basis for V is a linearly independent subset that spans V. To ensure linear independence we want to find the smallest number of vectors we can that span V.

The condition $a + d = 0$ gives $d = -a$, so any matrix in V can be written in the form

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

for some $a, b, c \in \mathbb{R}$. This can be rewritten as

$$\begin{aligned} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} &= \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \\ &= a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Which is a linear combination of three matrices. The set of these matrices

$$\{A, B, C\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

will be our proposed basis. To prove this is a basis we need to show three things:

(i) $\{A, B, C\}$ is a subset of V: This is easy to check. For the matrix A, we have $a = 1$ and $d = -1$, so $a + d = 0$. For matrices B and C, $a = 0$ and $d = 0$, so again $a + d = 0$.

(ii) $\{A, B, C\}$ spans V (i.e., every element of V is a linear combination of A, B, and C): We saw above that any element of V can be written as

$$aA + bB + cC$$

which is a linear combination of A, B, and C.

(iii) $\{A, B, C\}$ is linearly independent: Set an arbitrary linear combination equal to 0:

$$\begin{gathered} aA + bB + cC = \mathbf{0}_{2\times 2} \\ \downarrow \\ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{gathered}$$

where $a, b, c \in \mathbb{R}$. Looking at each matrix coordinate separately, we find that $a = 0$, $b = 0$, and $c = 0$. Therefore the set $\{A, B, C\}$ is linearly independent.

Thus all of the requirements for a basis are satisfied by $\{A, B, C\}$.

4. Are the functions $y_1 = \sin(x)$, $y_2 = \cos(x)$, and $y_3 = \sin(x+1)$ linearly independent over $\mathbb{R}$? Verify your answer.

   Solution: No. The functions are linearly dependent. To prove this, we need to find a nontrivial linear combination that is equal to zero. That is, we need to find numbers $c_1, c_2, c_3 \in \mathbb{R}$ (at least one of which is nonzero) such that

   $$c_1 \sin(x) + c_2 \cos(x) + c_3 \sin(x+1) = 0.$$

   for all $x \in \mathbb{R}$. The trick is to remember the sum-of-angles trig. identity:

   $$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta.$$

   Applying this to $\sin(x+1)$, we get

   $$\sin(x+1) = \sin(x)\cos(1) + \cos(x)\sin(1).$$

   Subtracting $\sin(x+1)$ from both sides gives

   $$0 = \cos(1)\sin(x) + \sin(1)\cos(x) - \sin(x+1)$$

   Now $\cos(1)$ and $\sin(1)$ are just constants, so this is a nontrivial linear combination that is equal to zero ($c_1 = \cos(1)$, $c_2 = \sin(1)$, and $c_3 = -1$). Therefore these functions are linearly dependent.