Homework 1

Answer Key

The vector space axioms in this worksheet are labeled as you saw in lecture (A0–A4, S0–S2, and I1–I2). Please see the solution to problem 1.2(a) below for details.

1. Do Problem 1.2 on page 5. If you answer yes, verify your answer by checking each axiom of vector space. If you answer no, give a counterexample showing how one of the axioms is violated.

   1.2 Which of the following sets (with natural addition and multiplication by a scalar) are vector spaces? Justify your answer.

   (a) The set of all continuous functions on the interval $[0, 1]$;

   Solution: This set is a vector space. Let's call this set V. The elements of V are continuous functions from $[0, 1]$ to $\mathbb{R}$, and the "natural" addition and scalar multiplication of functions is pointwise. That is, the functions $(f + g)$ and (αf) evaluated at the point $x \in [0, 1]$ are given by

$$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (\alpha f)(x) = \alpha f(x).$$

   It is important to note that **this set is closed under addition and scalar multiplication (axioms A0 and S0 are satisfied).**

   To prove that V is a vector space, we must verify each of the remaining vector space axioms.

   1. (A1) Addition is commutative: This follows directly from commutativity of addition in $\mathbb{R}$. Let $f, g \in V$. Then

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ &= g(x) + f(x) \qquad \text{by commutativity of addition in } \mathbb{R} \\ &= (g + f)(x)\end{aligned}$$

   so $f + g = g + f$.

   2. (A2) Addition is associative: Similarly to axiom 1, this follows directly from associativity of addition in $\mathbb{R}$. Let $f, g, h \in V$. Then

$$\begin{aligned}((f + g) + h)(x) &= (f + g)(x) + h(x) \\ &= (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) \qquad \text{by associativity of addition in } \mathbb{R} \\ &= f(x) + (g + h)(x) \\ &= (f + (g + h))(x),\end{aligned}$$

   so $(f + g) + h = f + (g + h)$.

   3. (A3) Zero vector: Let $z(x) = 0$ for all $x \in [0, 1]$. Then z is a continuous function from $[0, 1]$ to $\mathbb{R}$, so $z \in V$. Now let f be any element of V. Then

$$\begin{aligned}(f + z)(x) &= f(x) + z(x) \\ &= f(x) + 0 \\ &= f(x),\end{aligned}$$

   so $f + z = f$. Therefore $z = \mathbf{0}$.

4. **(A4) Additive inverse:** Let $f \in V$ and let z be the zero vector (as defined in the proof of axiom 3). Define g by $g(x) = -f(x)$ for all $x \in [0,1]$. Then $g \in V$ and

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ &= f(x) + (-f(x)) \\ &= 0 \\ &= z(x).\end{aligned}$$

Therefore $f + g = z = \mathbf{0}$, so g is the additive inverse of f.

5. **(S1) Multiplicative identity:** Let $f \in V$. Then

$$(1f)(x) = 1f(x) = f(x),$$

so $1f = f$.

6. **(S2) Scalar multiplication is associative:** Let $f \in V$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}((\alpha\beta)f)(x) &= (\alpha\beta)f(x) \\ &= \alpha(\beta f(x)) \qquad \text{by associativity of multiplication in } \mathbb{R} \\ &= \alpha(\beta f)(x) \\ &= (\alpha(\beta f))(x),\end{aligned}$$

so $(\alpha\beta)f = \alpha(\beta f)$.

7. **(I1) Scalar distribution 1:** Let $f, g \in V$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}(\alpha(f+g))(x) &= \alpha(f+g)(x) \\ &= \alpha(f(x) + g(x)) \\ &= \alpha f(x) + \alpha g(x) \qquad \text{by distribution in } \mathbb{R} \\ &= (\alpha f)(x) + (\alpha g)(x) \\ &= (\alpha f + \alpha g)(x),\end{aligned}$$

so $\alpha(f+g) = \alpha f + \alpha g$.

8. **(I2) Scalar distribution 2:** Let $f \in V$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}((\alpha+\beta)f)(x) &= (\alpha+\beta)f(x) \\ &= \alpha f(x) + \beta f(x) \qquad \text{by distribution in } \mathbb{R} \\ &= (\alpha f)(x) + (\beta f)(x) \\ &= (\alpha f + \beta f)(x),\end{aligned}$$

so $(\alpha + \beta)f = \alpha f + \beta f$.

(b) The set of all non-negative functions on the interval $[0,1]$;

Solution: This set is not a vector space. You are only required to give a single violation to the definition of a vector space. However, this answer key will detail which parts of the definition are satisfied and which parts are violated.

Call this set V. We can see that V is not a vector space, because it is **not closed under scalar multiplication (axiom S0 is not satisfied).** Let $f(x) = 1$ for all $x \in [0,1]$. Then $f \in V$, but $(-1)f(x) = -1$ for all $x \in [0,1]$, so $(-1)f \notin V$ (since it is not non-negative).

V is closed under addition (axiom A0 is satisfied). If $f, g \in V$ then $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in [0,1]$ so

$$(f+g)(x) = f(x) + g(x) \geq 0 \qquad \text{for all } x \in [0,1].$$

Therefore $f + g \in V$.

Now we move on to the other axioms:

- **Axioms A1, A2, A3, and S1 are satisfied.** The proof is the same as in 1.2(a).
- **Axiom A4 is not satisfied.** Let $f(x) = 1$ for all $x \in [0,1]$. If g is the additive inverse of f, then $g(x) = -1$ for all $x \in [0,1]$, but then $g \notin V$. Therefore $f \in V$ does not have an additive inverse.
- **Axioms S2, I1, and I2** assume that V is closed under scalar multiplication. Since V is not closed under scalar multiplication, these axioms are **not satisfied.**

(c) The set of all polynomials of degree exactly n;

Solution: This set is not a vector space (unless $n = 0$).

Assume $n > 0$ is a fixed natural number and let V be the set of all polynomials of degree exactly n. This set is **not closed under addition or scalar multiplication (axioms A0 and S0 are not satisfied),** so it cannot be a vector space:

- $f(x) = x^n$ and $g(x) = -x^n$ are degree n polynomials, but $f(x)+g(x) = 0$ is a degree 0 polynomial, so V is not closed under addition.
- $f(x) = x^n$ is a degree n polynomial, but $0f(x) = 0$ is a degree 0 polynomial, so V is not closed under scalar multiplication.

Now consider the other axioms:

- **Axioms A1, A2, and I1** assume that V is closed under addition. Since it is not, these axioms are **not satisfied.**
- **Axiom A3 is not satisfied.** The additive identity for polynomials is $z(x) = 0$, but this is a degree 0 polynomial, so $z(x) \notin V$.
- **Axiom A4** requires the existence of a zero vector (i.e., it requires that axiom A3 is satisfied). Since axiom A3 is not satisfied, axiom A4 is also **not satisfied**.
- **Axiom S1 is satisfied.** Let f be any polynomial. Then

  $$(1f)(x) = 1f(x) = f(x),$$

  so $1f = f$.
- **Axioms S2, I1, and I2** assume that V is closed under scalar multiplication. Since it is not, these axioms are **not satisfied.**

In the special case $n = 0$, V is just the set of real numbers with standard addition and scalar multiplication. It is easy to check that this is a vector space.

(d) The set of all symmetric $n \times n$ matrices, i.e. the set of matrices $A = \{a_{j,k}\}_{j,k=1}^{n}$ such that $A^T = A$.

Solution: This set is a vector space. Let's call this set V. If we consider these to be real matrices, then the scalars are assumed to be in $\mathbb{R}$. If we consider the matrices to be complex, then the scalars are assumed to be in $\mathbb{C}$. The proofs of the axioms work the same way in either case.

It is not too difficult to verify that **V is closed under addition and scalar multiplication (axioms A0 and S0 are satisfied).** It follows from these two easily verified facts:

- $(A+B)^T = A^T + B^T$ for any $n \times n$ matrices A and B.
- $(kA)^T = kA^T$ for any $n \times n$ matrix A and scalar k.

Now we check the axioms. Each proof is done by considering an arbitrary coordinate in the matrix. Recall that the coordinate in the ith row and jth column of a matrix A is denoted by $A_{i,j}$. Two matrices A and B of the same dimensions are equal when $A_{i,j} = B_{i,j}$ for all rows i and all columns j.

1. **(A1) Addition is commutative:** Let $A, B \in V$. Then

$$\begin{aligned} (A+B)_{i,j} &= A_{i,j} + B_{i,j} \\ &= B_{i,j} + A_{i,j} \qquad \text{by commutativity of addition in } \mathbb{R} \text{ or } \mathbb{C} \\ &= (B+A)_{i,j} \end{aligned}$$

so $A + B = B + A$.

2. **(A2) Addition is associative:** Let $A, B, C \in V$. Then

$$\begin{aligned} ((A+B)+C)_{i,j} &= (A_{i,j} + B_{i,j}) + C_{i,j} \\ &= A_{i,j} + (B_{i,j} + C_{i,j}) \qquad \text{by associativity of addition in } \mathbb{R} \text{ or } \mathbb{C} \\ &= (A+(B+C))_{i,j} \end{aligned}$$

so $(A+B)+C = A+(B+C)$.

3. **(A3) Zero vector:** Let Z be the $n \times n$ matrix where every coordinate has value 0. Then Z is symmetric, so $Z \in V$. Now let A be any element of V. Then

$$\begin{aligned} (A+Z)_{i,j} &= A_{i,j} + Z_{i,j} \\ &= A_{i,j} + 0 \\ &= A_{i,j}, \end{aligned}$$

so $A + Z = A$. Therefore Z is the zero vector in V.

4. **(A4) Additive inverse:** Let $A \in V$ and define B by $B_{i,j} = -A_{i,j}$. Then $B = (-1)A \in V$ since V is closed under scalar multiplication. Then

$$A_{i,j} + B_{i,j} = A_{i,j} - A_{i,j} = 0$$

so $A + B = Z$ where Z is the zero vector from the proof of axiom 3.

5. **(S1) Multiplicative identity:** Let $A \in V$. Then

$$(1A)_{i,j} = 1A_{i,j} = A_{i,j},$$

so $1A = A$.

6. (S2) Scalar multiplication is associative: Let α, β be scalars and let $A \in V$. Then

$$\begin{aligned}((\alpha\beta)A)_{i,j} &= (\alpha\beta)A_{i,j} \\ &= \alpha(\beta A)_{i,j} && \text{by associativity of multiplication in } \mathbb{R} \text{ or } \mathbb{C} \\ &= (\alpha(\beta A))_{i,j}\end{aligned}$$

7. (I1) Scalar distribution 1: Let α be a scalar and let $A, B \in V$. Then

$$\begin{aligned}(\alpha(A+B))_{i,j} &= \alpha(A_{i,j} + B_{i,j}) \\ &= \alpha A_{i,j} + \alpha B_{i,j}) && \text{by distribution in } \mathbb{R} \text{ or } \mathbb{C} \\ &= (\alpha A + \alpha B)_{i,j}),\end{aligned}$$

so $\alpha(A+B) = \alpha A + \alpha B$.

8. (I2) Scalar distribution 2: Let α, β be scalars and let $A \in V$. Then

$$\begin{aligned}((\alpha+\beta)A)_{i,j} &= (\alpha+\beta)A_{i,j} \\ &= \alpha A_{i,j} + \beta A_{i,j}) && \text{by distribution in } \mathbb{R} \text{ or } \mathbb{C} \\ &= (\alpha A + \beta A)_{i,j}),\end{aligned}$$

so $(\alpha+\beta)A = \alpha A + \beta A$.

2. Do Problem 1.7 on page 5. Make sure to write your arguments coherently in full sentences.

1.7 Prove that $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v} \in V$.

Solution: Let $\mathbf{v}$ be any vector in V. Then

$$\begin{aligned}0\mathbf{v} &= (0+0)\mathbf{v} \\ &= 0\mathbf{v} + 0\mathbf{v} && \text{(by axiom S2: scalar distribution).}\end{aligned}$$

Now let $-(0\mathbf{v})$ be the additive inverse of $0\mathbf{v}$ (the additive inverse exists by axiom A4). Adding $-(0\mathbf{v})$ to both sides of the above equation gives

$$\begin{aligned}0\mathbf{v} + [-(0\mathbf{v})] &= (0\mathbf{v} + 0\mathbf{v}) + [-(0\mathbf{v})] \\ &\downarrow \text{ (axiom A2: associativity of addition)} \\ 0\mathbf{v} + [-(0\mathbf{v})] &= 0\mathbf{v} + (0\mathbf{v} + [-(0\mathbf{v})]) \\ &\downarrow \text{ (axiom A4: additive inverse)} \\ \mathbf{0} &= 0\mathbf{v} + \mathbf{0} \\ &\downarrow \text{ (axiom A3: zero vector)} \\ \mathbf{0} &= 0\mathbf{v}.\end{aligned}$$

3. Do Problem 1.8 on page 5. Make sure to write your arguments coherently in full sentences.

   1.8 Prove that for any vector $\mathbf{v}$ its additive inverse $-\mathbf{v}$ is given by $(-1)\mathbf{v}$.

   Solution: Let $\mathbf{v}$ be any vector in V. To show that $(-1)\mathbf{v}$ is the additive inverse of $\mathbf{v}$ we must verify that $\mathbf{v} + (-1)\mathbf{v} = \mathbf{0}$.

$$\begin{aligned}
\mathbf{v} + (-1)\mathbf{v} &= 1\mathbf{v} + (-1)\mathbf{v} && \text{(by axiom S1: multiplicative identity)} \\
&= (1 + (-1))\mathbf{v} && \text{(by axiom I2: scalar distribution)} \\
&= (0)\mathbf{v} && \text{(simplification in } \mathbb{R} \text{ or } \mathbb{C}\text{)} \\
&= \mathbf{0} && \text{(by problem 1.7)}
\end{aligned}$$

4. Consider a set $V = \mathbb{R}$ and the field of real numbers $F = \mathbb{R}$. The addition on V is the usual addition of real numbers. But the scalar multiplication is defined differently as follows: $a * b = 2ab$ for all $a \in F$ and $b \in V$. Here the multiplication on the right hand side is the usual multiplication of real numbers. Is V, with operations $+$ and $*$, a vector space over F? Verify your answer.

   Solution: This is not a vector space. It is not hard to see that **V is closed under addition and scalar multiplication (axioms A0 and S0 are satisfied).** Since addition in V is defined as addition in $\mathbb{R}$, axioms A1–A4 are all satisfied. Axioms S1, S2, I1, and I2 all involve scalar multiplication. Since we are using non-standard scalar multiplication, this is where a violation might occur.

   - **Axiom S1 is not satisfied.** consider $5 \in V = \mathbb{R}$ and $1 \in F = \mathbb{R}$. Then

$$1 * 5 = 2(1)(5) = 10 \neq 5$$

   - **Axiom S2 is not satisfied.** Consider $5 \in V$ and $1 \in F$. We will simplify $(1 \cdot 1) * 5$ and $1 * (1 * 5)$ and compare the results:

$$\begin{aligned}
(1 \cdot 1) * 5 &= 1 * 5 && \text{since } 1 \cdot 1 = 1 \text{ in } F \\
&= 2(1)(5) \\
&= 10.
\end{aligned}$$

$$\begin{aligned}
1 * (1 * 5) &= 1 * 10 && \text{since } 1 * 5 = 10 \text{ in } V \\
&= 2(1)(10) \\
&= 20.
\end{aligned}$$

     Therefore

$$(1 \cdot 1) * 5 = 10 \neq 20 = 1 * (1 * 5).$$

   - **Axiom I1 is satisfied.** Let $\alpha \in F$ and $x, y \in V$. Then

$$\begin{aligned}
\alpha * (x + y) &= 2(\alpha)(x + y) \\
&= 2\alpha x + 2\alpha y \\
&= \alpha * x + \alpha * y.
\end{aligned}$$

   - **Axiom I2 is satisfied.** Let $\alpha, \beta \in F$ and $x \in V$. Then

$$\begin{aligned}
(\alpha + \beta) * x &= 2(\alpha + \beta)(x) \\
&= 2\alpha x + 2\beta x \\
&= \alpha * x + \beta * x.
\end{aligned}$$