FINITE HURWITZ BRAID GROUP ACTIONS ON SEQUENCES OF EUCLIDEAN REFLECTIONS

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Abstract. We prove that if $r_1, \dots, r_n$ are Euclidean reflections corresponding to a linearly independent set of vectors, then the group $< r_1, \dots, r_n >$ is finite if and only if the natural Hurwitz braid group action on such ordered sets of reflections has finite orbit and we characterise the orbits for this action. We apply this to give a representation of the braid group on n strands onto the alternating or symmetric groups of degree $(n+1)^{n-2}$ (for most n) which is relevant to the Morse theory of polynomials, as studied by Catanese, Paluszny and Wajnryb.

§1. Introduction

A *left self-distributive system* (LD-system) [De] is a set U, together with an action of the set on itself, satisfying : $a(bc) = (ab)(ac)$. A fundamental example of such an LD-system is when U is a group and the action is by conjugation. This was first considered by Hurwitz [Hur] and has applications to Galois theory [Vo]. Each such LD-system gives rise to a representation of the braid group $\mathcal{B}_n$. Here $\mathcal{B}_n$ is the group with (standard) generators $\sigma_1, \dots, \sigma_{n-1}$ and relations [Bi, p. 18]:

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{for } i = 1, \dots, n-2, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i, \qquad\quad \text{for } |i-j| > 1. \end{aligned}$$

Given $n \geq 2$ and an LD-system U there is an action of the braid group $\mathcal{B}_n =< \sigma_1, \dots, \sigma_{n-1} >$ on the product U^n where, in the conjugation case, the braid generator $\sigma_i, 1 \leq i < n$, acts as follows:

$$\begin{aligned} \sigma_i(s_1, s_2, \dots, s_n) &= (s_1, \dots, s_{i-1}, s_{i+1}, s_{i+1}^{-1}s_is_{i+1}, s_{i+2}, \dots, s_n) \\ \sigma_i^{-1}(s_1, s_2, \dots, s_n) &= (s_1, \dots, s_{i-1}, s_is_{i+1}s_i^{-1}, s_i, s_{i+2}, \dots, s_n). \end{aligned} \tag{1.1}$$

We will be interested in the case where U is a finite reflection group and $(s_1, s_2, \dots, s_n)$ is a minimal set of generating reflections.

Recall that to every vector $v \in \mathbb{R}^n, v \neq (0, \dots, 0)$, there is an associated reflection r_v whose action on $\mathbb{R}^n$ is given by:

$$r_v(w) = w - 2\frac{(v,w)}{\sqrt{(v,v)(w,w)}}v.$$

In what follows we will adopt the standard terminology for finite reflection groups ([GB, Hu]). Thus an irreducible finite reflection group $W =< S >$ will have a certain type X_n where $n = |S|$ is the minimal number of generating reflections. We prove:

Theorem 1.1. *Let $n \geq 3, v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ with $\|v_i\| = 1$ for $i = 1, \ldots, n$ and $v_1, v_2, \ldots, v_n$ linearly independent. Then the $\mathcal{B}_n$-orbit of $(r_{v_1}, r_{v_2}, \ldots, r_{v_n})$ is finite if and only if the group $< r_{v_1}, r_{v_2}, \ldots, r_{v_n} >$ is finite.*

The proof of the above result includes an analysis of the $\mathcal{B}_n$-orbits of the irreducible finite reflection groups W. In fact it is most natural to define the equivalence relation generated by the $\mathcal{B}_n$ action together with the action of W by conjugation on such n-tuples of reflections in W. We call this $\mathcal{BC}_n$-equivalence. This has been considered by Brieskorn [Br] in relation to singularities.

One may wonder whether, for a generating sequence $S = (s_1, s_2, \ldots, s_n)$ of a group G, the fact that the $\mathcal{B}_n$-orbit of S is finite is enough to be able to conclude that G is finite. This however is not the case, since any infinite group generated by a finite union of finite conjugacy classes will give a counterexample to this idea. For example, take a finite group H generated by a conjugacy class $C = \{c_1, \ldots, c_n\}$ and such that there is a relation $r(c_1, \ldots, c_n)$ in these generators having exponent 1 (for example a non-abelian finite simple group). Let $G = H \times \mathbb{Z}$. If $\mathbb{Z} =< t >$, then the elements $tC = \{tc_1, \ldots, tc_n\} \subset G$ form a finite conjugacy class in G and since $r(tc_1, \ldots, tc_n) = t$ we see that they generate G. Thus the $\mathcal{B}_n$ orbit of $(tc_1, \ldots, tc_n)$ is contained in $(tC)^n$ and so is finite.

For $S = (s_1, s_2, \ldots, s_n)$ as above we define the *Coxeter element* $\Pi(S)$ *of* S to be the product $s_1 s_2 \ldots s_n$. The *Coxeter element of a finite reflection group* W is $\Pi(S)$ where S is a base of W (defined below). Note from (1.1) that the Coxeter element of S is invariant under the $\mathcal{B}_n$ action. Further, the conjugacy class of the Coxeter element is not changed under $\mathcal{BC}_n$-equivalence. We will let $\mathcal{BC}_n(S)|$ denote the number of $\mathcal{BC}_n$ classes of S.

To establish notation we here give the families of finite reflection groups [GB]:

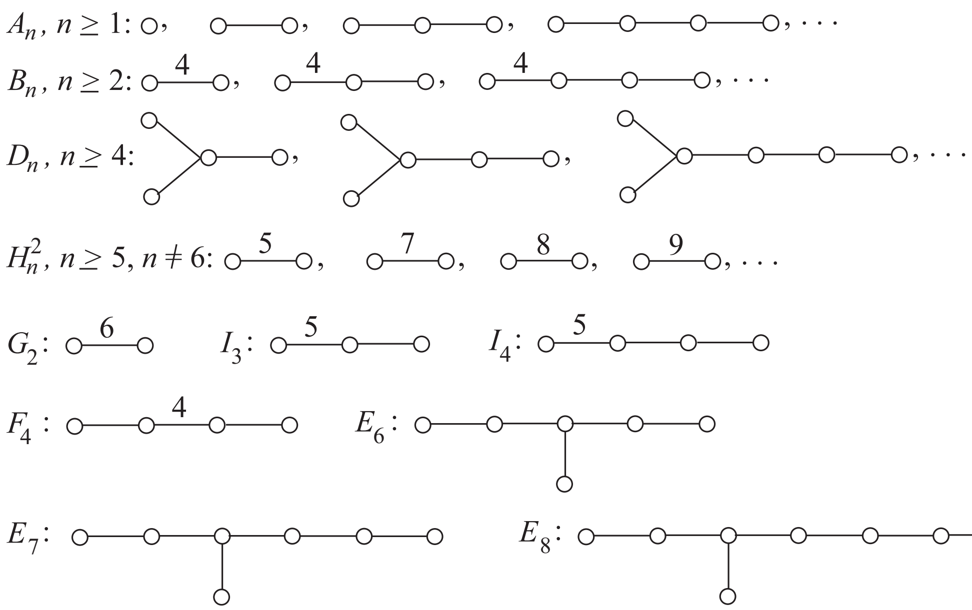

Figure 1.

We prove:

Theorem 1.2. *Let $n \geq 1$ and let $W =< S >$ be a finite reflection group. If W has type A_n or B_n, then there is one $\mathcal{BC}_n$ equivalence class of n reflections generating W.*

If W has type $D_n, n \geq 4$, then $|\mathcal{BC}_n(D_n)| = n - 2$. These are distinguished by their Coxeter elements.

If W has type I_3, then $|\mathcal{BC}_3(I_3)| = 3$. These are distinguished by their Coxeter elements.
If W has type I_4, then $|\mathcal{BC}_4(I_4)| = 11$. These are distinguished by their Coxeter elements.
If W has type F_4, then $|\mathcal{BC}_4(F_4)| = 2$. These are distinguished by their Coxeter elements.
If W has type E_6, then $|\mathcal{BC}_6(E_6)| = 3$. These are distinguished by their Coxeter elements.
If W has type E_7, then $|\mathcal{BC}_7(E_7)| = 5$. These are distinguished by their Coxeter elements.
If W has type E_8, then there are at least 9 $\mathcal{BC}_8$ classes which are distinguished by their Coxeter elements.

We will later give specific representatives for the $\mathcal{BC}_n$ classes.

Combining the above with a result of Howlett [Ho] (see also [Co; Theorem 12]) we have:

Corollary 1.3. *Let $n \geq 3$. Let $W = < r_1, \dots, r_n >$, where $r_i = r_{v_i}$ with $v_1, \dots, v_n$ linearly independent. Then the following are equivalent:*
(i) W is finite;
(ii) the product $r_1 \dots r_n$ is finite;
(iii) the $\mathcal{B}_n$ orbit of $(r_1, \dots, r_n)$ is finite.

The action of $\mathcal{B}_n$ on the set of sequences S of reflections generating finite groups gives interesting finite representations of the braid groups. Let $O_n(S)$ denote the $\mathcal{B}_n$-orbit of such an $S, |S| = n, | < S > | < \infty$, and let $PO_n(S)$ denote the set of equivalence classes in $O_n(S)$ determined by conjugacy. Then B_n acts on $PO_n(S)$ and we have permutation representations:

$$\mathcal{B}_n \to Sym(O_n(S)) \to Sym(PO_n(S)).$$

Let $\rho_n : \mathcal{B}_n \to Sym(PO_n(S))$ denote the composition.

We now consider the above representation in each of the cases A_n, B_n etc. The A_n case has connections with the work of Catanese, Paluszy and Wajnryb on the space of generic polynomials [CP, CW]; see also [Lo]. We also note that the isotropy subgroup for the action of $\mathcal{B}_n$ on a Coxeter system of an irreducible finite reflection group is determined in [Do].

If $< S >= W(A_n) \cong S_{n+1}$, then we will show that $PO_n(S)$ can be naturally identified with the set E_n of trees with n labeled edges (with vertices not labeled). There are $(n+1)^{n-2}$ of these [SSV, S]. It was this action of $\mathcal{B}_n$ on E_n that has been studied in connection with certain questions about the Morse theory of polynomials in [CW, CP, Lo].

Theorem 1.4. *Consider the $\mathcal{B}_n$ action on $PO_n(S)$ where $< S >= S_{n+1}$. Then the action gives a representation of $\mathcal{B}_n$ of degree $(n+1)^{n-2}$ which is 2-transitive for all $n \geq 3$. If $(n+1)^{n-2}$ is not of the form*

$$q, \ \frac{q^d - 1}{q - 1} \ (\text{for } d \geq 2), \ \ 2^{d-1}(2^d \pm 1) \ (\text{for } d \geq 3),$$

where q is a prime power, then the $\mathcal{B}_n$ action is that of the symmetric or alternating group of degree $(n+1)^{n-2}$. Specifically, if $n > 3$ and $(n+1)^{n-2}$ is not of the above form, then we get the alternating group $A_{(n+1)^{n-2}}$ when $n \equiv 1, 2, 3 \ mod \ 4$ and the symmetric group $S_{(n+1)^{n-2}}$ otherwise.

The above result uses the classification of 2-transitive permutation groups [Cam], which, in turn, depends on the classification of finite simple groups.

We now describe the results of [CP], [CW]. Let $P \in \mathbb{C}[z]$ be a polynomial of degree $n+1$. Then P is *lemniscate generic* if, when $y_1, \dots, y_n$ are the roots of the derivative $P'(z)$, we

have (i) $P(y_i) \neq 0$ and (ii) $|P(y_i)| = |P(y_j)|$ only when $i = j$. Let $\mathcal{L}_n$ be the set of such, considered as a subset of $\mathbb{C}^{n+1}$, the space of all polynomials of degree $n+1$. To $P \in \mathcal{L}_n$ we associate the *big lemniscate configuration*, this being the union of the singular level sets of $|P|$, considered up to homeomorphism. This is invariant as P moves in a connected component of $\mathcal{L}_n$. Let E_n be the set of isomorphism classes of edge-labeled trees on n edges. Then [CP, CW] there is a natural mapping from E_n to the set of big lemniscate configurations. There is also a natural action of $\mathcal{B}_n$ on E_n; the same as constructed above. Let $\Lambda_n =< (\sigma_1^2, (\sigma_2\sigma_1)^3, (\sigma_3\sigma_2\sigma_1)^4, \dots, (\sigma_{n-1}\dots\sigma_2\sigma_1)^n >\cong \mathbb{Z}^{n-1}$. Then the main result of [CP] is: There is a natural bijection between:
(i) $\pi_0(\mathcal{L}_n)$;
(ii) The set of big lemniscate configurations;
(iii) The set of Λ_n-orbits on E_n;
(iv) The set

$$\{(x_0, x_1, \dots, x_n) | x_i \in \mathbb{N}, 0 \leq x_i \leq i, \text{ no integer occurs three times in the sequence}\}.$$

In our notation E_n is just $PO_n(S)$, where S is of type A_n.

We recall [Bi] the natural permutation representation $\mathcal{B}_n \to S_n$ given by $\sigma_i \mapsto (i, i+1)$. For the B_n case we have:

Theorem 1.5. *Consider the $\mathcal{B}_n$ action on $PO_n(S)$, $n \geq 3$, where $< S >$ is of type B_n. Then this action gives a representation of $\mathcal{B}_n$ of degree n^{n-1} which is transitive. The action is not primitive and there is a block system, having n blocks, such that the action of $\mathcal{B}_n$ on the blocks gives the permutation representation $\pi_n : \mathcal{B}_n \to S_n$.*

The action in the D_n case is more complicated. For the remaining cases we use MAGMA [MA] to get the following result, where we also indicate some of the representatives of the different $\mathcal{BC}_n$ classes:

Theorem 1.6. ($\mathbf{I_3}$) *Let $G =< a, b, c | a^2, b^2, c^2, (ab)^5, (bc)^3, (ac)^2 >$ be of type I_3. There are three $\mathcal{BC}_3$ orbits with the following representatives:*
(i) $S = (a, b, c)$. Then $|PO_3(S)| = 10$ and $\rho_3(\mathcal{B}_3) = S_{10}$.
(ii) $S = (a, b, cbacbcabc)$. Then $|PO_3(S)| = 18$ and the $\mathcal{B}_3$ action gives a transitive but imprimitive representation into S_{18}. There is a unique set of blocks of size two and the action on these blocks is that of the alternating group A_9. The kernel is $\mathbb{Z}_2^9$.
(iii) $S = (a, c, cbacbcabc)$. Then $|PO_3(S)| = 10$ and $\rho_3(\mathcal{B}_3) = S_{10}$.

($\mathbf{I_4}$) *Let $G =< a, b, c, d | a^2, b^2, c^2, d^2, (ab)^5, (bc)^3, (ac)^2, (ad)^2, (bd)^2, (cd)^3 >$ be of type I_4. There are 11 $\mathcal{BC}_4$ orbits with representatives:*
(i) $S = (a, b, c, d)$. Then $|PO_4(S)| = 90$ and $\rho_4(\mathcal{B}_4) = S_{90}$.
(ii) $S = (a, b, c, babcdcbab)$. Then $|PO_4(S)| = 10$ and $\rho(\mathcal{B}_4) = S_{10}$.
(iii) $S = (a, b, c, dcbabcdcbabcd)$. Then $|PO_4(S)| = 160$ and $\rho_4(\mathcal{B}_4) = A_{160}$.
(iv) $S = (a, b, c, dcababcdcbabacd)$. Then $|PO_4(S)| = 135$ and $\rho_4(\mathcal{B}_4) = S_{135}$.
(v) $S = (a, b, c, dcbabcababcdcbabacbabcd)$. Then $|PO_4(S)| = 288$. The action is not primitive and has a unique block system with blocks of size two. The action on the 144 blocks gives a representation of $\mathcal{B}_4$ onto A_{144}.
(vi) $S = (a, b, c, babcdcbabacbabcdcbabcababcdcbab)$. Then $|PO_4(S)| = 90$. The action is not primitive and has a unique block system with blocks of size two. The action on the 45 blocks gives a representation of $\mathcal{B}_4$ onto S_{45}. The kernel of the action on the blocks is $\mathbb{Z}_2^{44}$.

(vii) $S = (a, b, acbabca, badcbabcdab)$. *Then* $|PO_4(S)| = 18$. *The action is not primitive and has a unique block system with blocks of size two. The action on the* 9 *blocks gives a representation of* $\mathcal{B}_4$ *onto* A_9. *The kernel of the action on the blocks is* $\mathbb{Z}_2^9$.
(viii) $S = (a, b, d, cbabacbadcbabcdabcababc)$. *Then* $|PO_4(S)| = 160$ *and* $\rho_4(\mathcal{B}_4) = A_{160}$.
(ix) $S = (a, b, cbabc, cdcbabacbabcdcbabcababcdc)$. *Then* $|PO_4(S)| = 135$ *and* $\rho_4(\mathcal{B}_4) = S_{135}$.
(x) $S = (a, c, dcbabcd, babacbabcdcbabcabab)$. *Then* $|PO_4(S)| = 90$ *and* $\rho_4(\mathcal{B}_4) = S_{90}$.
(xi) $S = (a, c, babcbab, ababcdcbabacbabcdcbabcababcdcbaba)$. *Then* $|PO_4(S)| = 10$ *and* $\rho_4(\mathcal{B}_4) = S_{10}$.

($\mathbf{F_4}$) *Let* $G =< a, b, c, d|a^2, b^2, c^2, d^2, (ab)^3, (bc)^4, (ac)^2, (ad)^2, (bd)^2, (cd)^3 >$ *be of type* F_4. *There are two* $\mathcal{BC}_4$ *orbits with the following representatives:*
(i) $S = (a, b, c, d)$. *Then* $|PO_4(S)| = 72$ *and* $\rho_4(\mathcal{B}_4)$ *is not primitive.*
(ii) $S = (a, b, c, bcdcb)$. *Then* $|PO_4(S)| = 18$ *and* $\rho_4(\mathcal{B}_4)$ *is not primitive.*

($\mathbf{E_6}$) *Let* $G = < a, b, c, d, e, f|a^2, b^2, c^2, d^2, e^2, f^2, (ab)^3, (bc)^3, (ac)^2, (ad)^2, (bd)^2, (cd)^3, (ae)^2, (af)^2, (be)^2, (bf)^2, (ce)^2, (cf)^3, (de)^3, (df)^2, (ef)^2 >$ *be of type* E_6. *There are three* $\mathcal{BC}_6$ *orbits:*
(i) $S = (a, b, c, d, e, f)$. *Then* $|PO_6(S)| = 3456$ *and* $\rho_6(\mathcal{B}_6) = A_{3456}$.
(ii) $S = (a, b, c, d, e, cfabcdedcbafc)$. *Then* $|PO_6(S)| = 1080$ *and* $\rho_6(\mathcal{B}_6) = A_{1080}$.
(iii) $S = (a, b, c, d, e, bacbedcfcdebcab)$. *Then* $|PO_6(S)| = 6561 = 3^8$ *and* $\rho_6(\mathcal{B}_6) = A_{6561}$.

($\mathbf{E_7}$) *Let* $G = < a, b, c, d, e, f, g|a^2, b^2, c^2, d^2, e^2, f^2, g^2, (ab)^3, (bc)^3, (ac)^2, (ad)^2, (bd)^2, (cd)^3, (ae)^2, (af)^2, (be)^2, (bf)^2, (ce)^2, (cf)^3, (de)^3, (df)^2, (ef)^2, (ag)^2, (bg)^2, (cg)^2, (dg)^2, (eg)^3, (fg)^2 >$ *be of type* E_7. *There are five* $\mathcal{BC}_7$ *orbits with the following representatives :*

(i) (a, b, c, d, e, f, g); (ii) $(a, b, c, d, e, f, cdegedc)$; (iii) $(a, b, c, d, e, f, cbfcdegedcfbc)$;
(iv) $(a, b, c, d, e, deged, cbdcgcdbc)$; (v) $(a, b, c, d, e, bdcfcdb, cbfcdegedcfbc)$.

($\mathbf{E_8}$) *Let* $G = < a, b, c, d, e, f, g, h >$ *be of type* E_8 *(relations as for* E_7*, with the only non-commutator relation involving* h *being* $(gh)^3$*). The following represent nine different* $\mathcal{BC}_8$ *classes, with non-conjugate Coxeter elements:*

(i) (a, b, c, d, e, f, g, h); (ii) $(a, b, c, d, e, f, g, cdeghgedc)$;
(iii) $(a, b, c, d, e, f, g, cbfcdeghgedcfbc)$; (iv) $(a, b, c, d, e, f, g, efcbadcbfcdeghgedcfbcdabcfe)$;
(v) $(a, b, c, d, e, f, adcbfcdeghgedcfbcda, h)$; (vi) $(a, b, c, d, e, f, cbfcdeghgedcfbc, cdegedc)$;
(vii) $(a, b, c, d, e, f, adcbfcdeghgedcfbcda, cbfcdegedcfbc)$;
$(xiii)$ $(a, b, c, d, e, gedcbfcdeghgedcfbcdeg, bdcfcdb, g)$;
(ix) $(d, c, e, b, bdcfcdb, defcbadcbfcdeghgedcfbcdabcfed, acbdcfcdbca, ege)$.

§2 The n odd case

In this section we prove Theorem 1.1 in the case where $n \geq 3$ is odd.

Since $v_1, v_2, \ldots, v_n$ are linearly independent we may use them as a basis; relative to this basis we see that each $r_i = r_{v_i}$ has the following form:

$$r_i = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & -1 & \dots & a_{in-1} & a_{in} \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{2.1}$$

Here $a_{ij} = -2(v_i, v_j) = -2cos\theta_{ij}$ and the non-zero off-diagonal entries of r_i occur in the ith row.

Given any sequence $S = (r_1, r_2, \dots, r_n)$ of reflections in $GL(n, \mathbb{R})$ we define the graph $\Gamma(S)$ to have n vertices w_i and an edge from w_i to w_j if r_i and r_j do not commute. We will assume that the graph $\Gamma(S)$ is connected, this condition corresponding to the irreducibility of the group $W =< S >$ [Hu].

Lemma 2.1. *(i) For $1 \le i < j \le n$ there is $\alpha \in \mathcal{B}_n$ such that $\alpha(r_1, \dots, r_n) = (\dots, r_i, r_j, \dots)$ (with r_i occurring in the ith position and r_j in the $(i+1)$th position).*
(ii) For $1 \le i_1 < i_2 < \dots < i_k \le n$ there is $\alpha \in \mathcal{B}_n$ such that

$$\alpha(r_1, \dots, r_n) = (\dots r_{i_1}, r_{i_2}, \dots, r_{i_k} \dots).$$

(with r_{i_1} occurring in the i_1th position and then $r_{i_1}, r_{i_2}, r_{i_3}, \dots$ occurring consecutively).
(iii) For $1 \le i_1 < i_2 < \dots < i_k \le n$ and $1 \le t \le n - r + 1$ there is $\alpha \in \mathcal{B}_n$ such that

$$\alpha(r_1, \dots, r_n) = (\dots, r_{i_1}, r_{i_2}, \dots, r_{i_k}, \dots).$$

(with r_{i_1} occurring in the tth position).
(iv) For all $(g_1, g_2, \dots, g_n) \in W^n$ we have

$$\sigma_1\sigma_2 \dots \sigma_{n-1}(g_1, g_2, \dots, g_n) = g_n^{-1}(g_n, g_1, g_2, \dots, g_{n-1})g_n.$$

Proof. (i) Since $\sigma_j(r_1, r_2, \dots, r_n) = (r_1, \dots, r_{j-1}, r_{j+1}, r_j^{r_{j+1}}, r_{j+2}, \dots, r_n)$, we can move r_j one position closer to r_i. Repeating this gives (i) and repeating (i) gives (ii). Now (iii) follows since we can first assume that we have (ii) and then move this sequence of elements up or down. For example the braid $\sigma_{n-1} \dots \sigma_{i_1+1}\sigma_{i_1}\sigma_{i_1-1}$ moves the sequence down by one. Lastly, (iv) is proved by induction on n. $\square$

Lemma 2.2. *For $1 \le i \ne j \le n$, each θ_{ij} is a rational multiple of 2π.*

Proof. Relative to the basis v_i, v_j the reflections r_i and r_j are represented as

$$\begin{pmatrix} -1 & 0 \\ a_{ij} & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a_{ij} \\ 0 & -1 \end{pmatrix},$$

(respectively), with $a_{ij} = -2cos(\theta_{ij})$. Now by Lemma 2.1 (i) we may assume that $j = i+1$. Now note that $\sigma_i^{2k}(r_i) = (r_i r_{i+1})^k r_i (r_i r_{i+1})^{-k}$ is a reflection corresponding to the vector

$(r_ir_{i+1})^k(v_i)$ and that $\sigma_i^{2k}(r_{i+1}) = (r_ir_{i+1})^k r_{i+1}(r_ir_{i+1})^{-k}$ is the reflection corresponding to the vector $(r_ir_{i+1})^k(v_{i+1})$.

We note that r_ir_{i+1} is a rotation by angle $2\theta_{ii+1}$. Thus if r_ir_{i+1} has infinite order, then the fact that v_i, v_{i+1} are linearly independent means that there are infinitely many of the vectors $\{(r_ir_{i+1})^k(v_i), (r_ir_{i+1})^k(v_{i+1})\}_k$, contradicting the fact that the $\mathcal{B}_n$ orbit is finite. □

Thus we may modify the graph $\Gamma(S), S = (r_1, \ldots, r_n)$, by attaching to the edge between the ith and the jth vertex the label m where r_ir_j has order m. Let $\Gamma(S)_{ij}$ denote this number.

Proposition 2.3. *Assume that $\Gamma(S)$ is connected, that $|S| > 1$ is odd and that the $\mathcal{B}_n$ orbit of S is finite. Then $< S >$ is an irreducible finite reflection group.*

Proof. Suppose that $S = (r_1, \ldots, r_n)$. Then $\Pi(S) = r_1r_2 \ldots r_n$. We will need a result of Howlett [Ho]. Recall that a Coxeter group is a group having a presentation of the form $< x_1, \ldots, x_n | x_i^2, (x_ix_j)^{m_{ij}}, 1 \leq i \neq j \leq n >$. Here, if we put $m_{ii} = 1$ for all $i \leq n$, the matrix (m_{ij}) is a symmetric with entries in $\{1, 2, 3, \ldots \infty\}$. Howlett's result says that an arbitrary Coxeter group is finite if and only if the Coxeter element $x_1x_2 \ldots x_n$ is of finite order. Now the group $< S >$ is a quotient of a Coxeter group. Thus we see that $< S >$ is finite if and only if $\Pi(S)$ has finite order. We will now show that $\Pi(S)$ has finite order.

Let $\Delta^2 = (\sigma_1\sigma_2 \ldots \sigma_{n-1})^n \in \mathcal{B}_n$. Then by Lemma 2.1 (iv) we have

$$\Delta^{2k}(r_1, \ldots, r_n) = \Pi(S)^k(r_1, \ldots, r_n)\Pi(S)^{-k}$$

for all $k \in \mathbb{Z}$. Since we are assuming that the $\mathcal{B}_n$-orbit of S is finite we see that there must be some $N \in \mathbb{N}$ such that

$$\Pi(S)^N r_i \Pi(S)^{-N} = r_i$$

for all $1 \leq i \leq n$. We may assume that N is even. This shows that $\Pi(S)$ must belong to the centre of $< S >$. But since $\Gamma(S)$ is connected it follows that $< S >$ is irreducible [Hu] and so $\Pi(S)$ must be a scalar matrix (by Schur's lemma). The next result shows that, since n is odd, $\Pi(S)^2$ has 1 as an eigenvalue and since N is even we see that the diagonal matrix $\Pi(S)^N$ must be the identity. Thus $\Pi(S)^2$ has finite order. We now show that $\Pi(S)^2$ has 1 as an eigenvalue. In fact we will show that $\Pi(S)$ has -1 as an eigenvalue.

If M is an $n \times n$ matrix, then by M_{ij} we mean the matrix obtained from M by deleting the ith row and the jth column.

Recall [A] that if $M = (m_{ij})$ is an $n \times n$ antisymmetric matrix with n even, then there is an integral polynomial $P = P(m_{ij})$ such that $det(M) = P^2$. Now let $M = (m_{ij})$ be an $n \times n$ antisymmetric matrix with n odd. Then for each $1 \leq i \leq n$ there are integer polynomials $p_i(m_{rs})$ such that $det(M_{ii}) = p_i^2$. The polynomials p_i are clearly determined up to sign. However we will need to assume that for $\alpha \in S_n$ we have $p_{\alpha(1)} = \alpha(p_1)$ and that in p_1 the coefficient of $m_{23}m_{45} \ldots m_{n-1n}$ is 1. This now determines the p_i uniquely. We will call $p_i = p_i(M)$ the *ith Pfaffian of M*.

In the next few results we will assume that the a_{ij} are algebraically independent over $\mathbb{Q}$ (any other case is just a quotient of this case). So we fix S with (2.1) giving the action of the reflections and defining the a_{ij}. Let $M = (a_{ij})$ be the $n \times n$ antisymmetric matrix

$$M = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots \\ -a_{12} & 0 & a_{23} & \cdots \\ -a_{13} & -a_{23} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It suffices now to prove:

Lemma 2.4. *Suppose that $n > 1$ is odd. Then $\Pi(S)$ has -1 as an eigenvalue. The corresponding eigenvector is*

$$(p_1, -p_2, p_3, -p_4, \dots, p_n)^T,$$

where p_i is the ith Pfaffian of M.

Proof. We will need the following result, where $adj(M)$ is the classical adjoint matrix of M:

Lemma 2.5. *Let $M = (a_{ij})$ be the antisymmetric $n \times n$ matrix defined above, where n is odd. Then*

$$adj(M) = (p_1, -p_2, \dots, -p_n)^T (p_1, -p_2, \dots, -p_n).$$

Proof. Recall that for any $n \times n$ matrix M we have $rank(adj(M)) = n, 1, 0$ depending on whether $rank(M) = n, n-1, r < n-1$. Since n is odd and M is a generic anti-symmetric matrix we have $det(M) = 0, rank(M) = n-1$ and so we see that $adj(M)$ is a rank 1 matrix and so has the form $v^T w$ for some n-vectors v, w. Now we note that the i, i entry of $adj(M)$ is the determinant of the matrix M_{ii}. This is p_i^2.

Next note that since M is antisymmetric and n is odd, then $adj(M)$ is a symmetric matrix. Let $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n)$. Then $adj(M)_{ii} = p_i^2$ implies that $v_i, w_i \neq 0$. Now v, w are determined up to a scalar multiple and so we may scale so that $v_1 = p_1$. Then we have $v_1 w_1 = p_1^2$ and so $w_1 = p_1$. Now for $1 < i \leq n$ we have $p_1 w_i = v_1 w_i = v_i w_1 = v_i p_1$ and so $w_i = v_i$. We also have $p_i^2 = w_i v_i = v_i^2$, showing that $v_i = w_i = \pm p_i$ for all $1 < i \leq n$. It remains to determine the sign. This we do by looking at the sign of some monomials occurring in the p_i.

Now p_1 has the term $m_{23} m_{45} \dots m_{n-1n}$ and p_2 has the term $m_{13} m_{45} \dots m_{n-1n}$. But in $det(M_{1,2})$ the term $m_{23} m_{13} (m_{45} \dots m_{n-1n})^2$ occurs and so in $adj(M)$ the $1, 2$ entry contains $-m_{23} m_{13} (m_{45} \dots m_{n-1n})^2$. Thus $v = (p_1, -p_2, \dots)$ and the rest follows similarly. □

Since $M adj(M) = det(M) I_n$, where I_n denotes the identity matrix if rank n, we now see that

$$M(p_1, -p_2, p_3, \dots, -p_n)^T = 0.$$

In particular for each $1 \leq r \leq n$ we have

$$\sum_{i=1}^{r-1} -a_{ir}(-1)^{i+1} p_i + \sum_{i=r+1}^{n} a_{ri}(-1)^{i+1} p_i = 0. \tag{2.2}$$

Let $I_{nk}, 1 \leq k \leq n$, be the $n \times n$ diagonal matrix $diag(1, 1, \dots, 1, -1, -1, \dots, -1)$ where there are $n+1-k$ of the -1s. Then $I_{nn+1} = I_n$ and we have

Lemma 2.6. *Let $n > 1$ be odd. Let r_i be given by (2.1) and let $p_i = p_i(M)$ denote the Pfaffians as constructed above and let $P = (p_1, -p_2, p_3, \dots, p_n)^T$. Then for all $1 \leq k \leq n$ we have*

$$r_k I_{nk+1} P = I_{nk} P.$$

In particular we have $r_1 r_2 \dots r_n(P) = -P$.

Proof. The last statement follows from the first since $I_{nn+1} = I_n, I_{n1} = -I_n$ and

$$\begin{aligned} r_1 \dots r_{n-2} r_{n-1} r_n(P) &= r_1 \dots r_{n-2} r_{n-1} r_n I_{n+1}(P) \\ &= r_1 \dots r_{n-2} r_{n-1} I_{nn}(P) \\ &= r_1 \dots r_{n-2} I_{nn-1}(P) \\ &= \dots = I_{n1} P = -P. \end{aligned}$$

The first statement is proved as follows. Let J_i be the $n \times n$ diagonal matrix

$$J_i = diag(1, \dots, 1, -1, 1, \dots, 1),$$

with the -1 in the ith position. Thus $I_{nk} = J_k I_{nk+1}$. Now $r_i = J_i + K_i$ defines the matrix K_i and we have $K_i I_{ni+1} P = 0$ by (2.2). Thus

$$\begin{aligned} r_k I_{nk}(P) &= (J_k + K_k) I_{nk}(P) \\ &= I_{nk-1}(P), \end{aligned}$$

as required. □

As explained above, this now completes the proof of Proposition 2.3 and of Theorem 1.1 if n is odd. □

Proposition 2.7. *Suppose that W is an irreducible finite reflection group generated by a set of reflections S. Then for $s, s' \in S, s \neq s'$, the order of ss' is one of* $2, 3, 4, 5$.

If $|S| \geq 5$, then for $s, s' \in S, s \neq s'$ the order of ss' is one of $2, 3, 4$.

Proof. This follows from the classification of irreducible finite reflection groups [GB] by considering each case separately:

If W has type A_n, then $W = S_{n+1}$ and the reflections are the transpositions (ij) and if $(ij) \neq (rs)$, the order of $(ij)(rs)$ is 2 or 3.

If W has type B_n, then W is the group of signed permutations and the reflections are the transpositions (ij) and the elements $\epsilon_i, i = 1, \dots, n$, where $\epsilon_i(i) = -i, \epsilon_i(j) = j, j \neq i$. One finds that the orders are $2, 3$ or 4.

When W has type D_n, then W is a subgroup of the B_n case and so this case follows from the last.

The remaining cases I_3, I_4, E_6, E_7, E_8 are each checked using MAGMA [MA]; the orders for I_3, I_4 are $2, 3, 5$, while those for E_6, E_7, E_8 are $2, 3$. □

Corollary 2.8. *Suppose that $n \geq 3$ and that $\Gamma(S)$ is connected. Then each $\Gamma(S)_{ij}$ is in the set $\{2, 3, 4, 5\}$. If $n \geq 5$ and $\Gamma(S)$ is connected, then each $\Gamma(S)_{ij}$ is in the set $\{2, 3, 4\}$.*

Proof. Since $n \geq 3$ and $\Gamma(S)$ is connected, any edge of $\Gamma(S)$ is in a full subgraph of $\Gamma(S)$ which has three vertices and is connected. The first result now follows from Lemma 2.1 together with Proposition 2.7. The second follows likewise by considering connected subgraphs with 5 vertices. □

§3 Braid groups and finite reflection groups

Recall [GB] that given an irreducible finite reflection group $W =< S >, S = (r_1, \dots, r_n)$, $r_i = r_{v_i}, v_i \in \mathbb{R}^n$, the set of *roots* is the set of all images of the elements of $v_1, \dots, v_n$ under W; this is denoted by Δ. We will let $R(\Delta)$ denote the set of all reflections in W. Further, we may partition $\Delta = \Delta^+ \cup \Delta^-$ so that $-\Delta^- = \Delta^+$. Then a *base* is a subset $\Pi \subset \Delta$ such that Π is a basis for $\mathbb{R}^n$ and such that for $v \in \Delta^+$ the coordinates of v relative to Π are all positive.

In this section we prove Theorem 1.2. This will be by a case-by-case analysis. In each case we will use a standard description of the finite reflection groups as a group of permutations; see [GB, §5.3] for example.

First consider type A_n. Here $W \cong S_{n+1}$ and we may take $R(\Delta)$ to be the set of transpositions in S_{n+1}. So we will deal with sequences $T = (t_1, \dots, t_n)$ of such transpositions. To such a sequence T we associate a graph $\Gamma(T)$ with vertices $1, 2, \dots, n+1$ and with an edge $\{i, j\}, j \neq i$ if $(i, j) \in \{t_1, \dots, t_n\}$. For type A_n any base corresponds to a T such that $\Gamma(T)$ is a line graph (a line graph being a graph isomorphic to $\Gamma(((1,2),(2,3),\dots,(k,k+1))))$ for some k. Thus we need to show that for any T as above there is $\alpha \in \mathcal{B}_n$ such that $\Gamma(\alpha(T))$ is a line graph.

In any case we note that $\Gamma(T)$ is a graph with $n+1$ vertices and n edges, which is connected since $< T >= S_{n+1}$, and so must be a tree. Conversely, we note that if T is a sequence of n transpositions such that $\Gamma(T)$ is connected, then $< T >= S_{n+1}$. For the hypothesis here easily shows that $< T >$ is a transitive subgroup of S_{n+1} and any transitive subgroup of S_{n+1} containing a transposition is S_{n+1}.

Now for $T = (t_1, \dots, t_n)$ we let $\Gamma^*(T)$ denote the graph $\Gamma(T)$ only where we label the edge $\{r, s\}$ by i if $t_i = (r, s)$. The proof will now consist of various steps. If $i \leq n$ and K is a subgraph of $\Gamma^*(S)$, then we will let K_i denote the edge of K that is labeled i. An edge is *free* if it is adjacent to no more than one other edge.

In Figure 2 we indicate how the generators $\sigma_k^{\pm 1}$ act on the graphs $\Gamma^*(T)$. In Figure 2 (i) we give the case where the edges labeled k and $k+1$ are disjoint. In Figure 2 (ii), (iii) we give the action of $\sigma_k^{\pm 1}$ in the case where these edges are adjacent in $\Gamma^*(T)$.

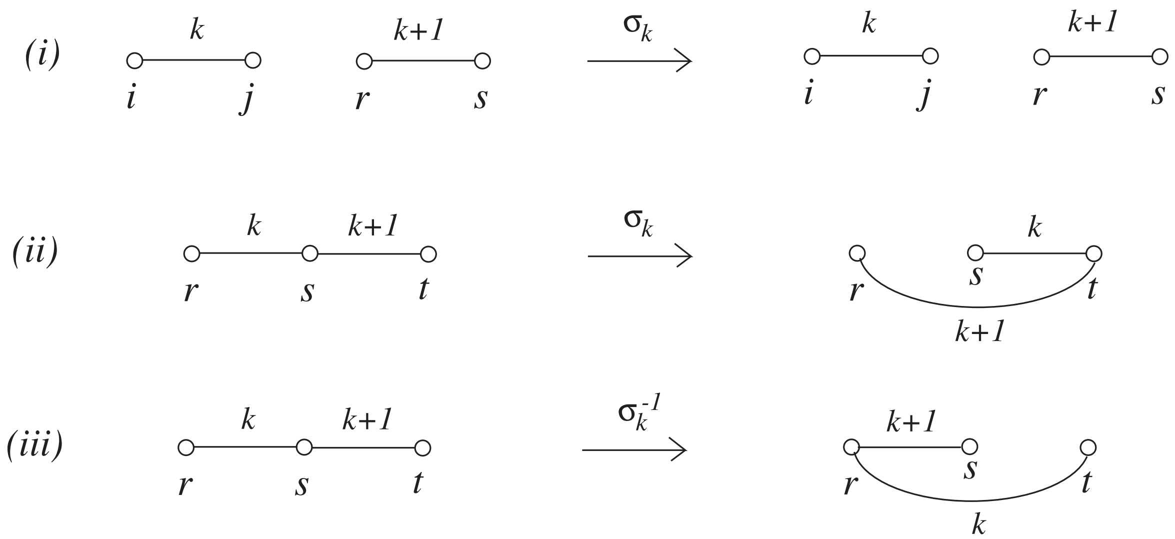

Figure 2.

In the following we will think of edges of $\Gamma(S)$ as being given by pairs $\{i, j\}$ of their vertices. Thus, as in the next result, it makes sense to think of an edge as belonging to both $\Gamma(S)$ and to $\Gamma(\alpha(S))$ for $\alpha \in \mathcal{B}_n$.

Lemma 3.1. *Suppose that L is a line subgraph of $\Gamma^*(T)$ with free edges $e_i = \Gamma^*(T)_i, e_j = \Gamma^*(T)_j, j \geq i$. Let $v \in e_j$ be an end vertex of L. Let W be the union of the closures of the connected components of $\Gamma^*(T) \setminus L$ that contain v and suppose that if $k < i$, then e_k is in W. Then there is $\alpha \in \mathcal{B}_n$ such that $\Gamma^*(\alpha(T))_i$ contains the vertex v and $\Gamma^*(\alpha(T))_r = e_r$ for every $r < i$.*

Proof. We will be inducting on q, the number of edges of L. If $q = 1$, then $i = j$ and we are done (take $\alpha = id$); so assume inductively that $q > 1$.

Let $e_m = \Gamma^*(T)_m$ be the unique edge of L which is adjacent to e_i. Then by hypothesis $m > i$. We now show how to reduce to the case where $m = i + 1$. If we have $m = i + 1$, then we act by σ_i^{-1} and we have reduced q by 1 (see Figure 2).

To reduce to the case $m = i + 1$ we assume that $m > i + 1$ and consider four cases:
a) $\Gamma^*(T)_m$ and $\Gamma^*(T)_{m-1}$ are disjoint;
b) not a) and $\Gamma^*(T)_{m-1}$ is an edge of L;
c) not a), not b), and all of $\Gamma^*(T)_i$, $\Gamma^*(T)_m$, $\Gamma^*(T)_{m-1}$ share a vertex;
d) $\Gamma^*(T)_i$ and $\Gamma^*(T)_{m-1}$ are disjoint, but $\Gamma^*(T)_m$ and $\Gamma^*(T)_{m-1}$ share a vertex.

If a), then σ_{m-1} has the effect of interchanging the labels on $\Gamma^*(T)_m$ and $\Gamma^*(T)_{m-1}$; thus we are done in this case.

If b), then we do σ_{m-1}^{-1}, which actually reduces the length of a line graph between $\Gamma^*(T)_i$ and $\Gamma^*(T)_j$.

For c) or d) we likewise do σ_{m-1}^{-1}. This proves Lemma 3.1. □

Now we describe the various steps in the proof of Theorem 1.2 in the A_n case:
Step 1: Find $\alpha \in \mathcal{B}_n$ such that $\Gamma^*(\alpha T)_1$ is a free edge of the tree $\Gamma^*(\alpha T)$.

If $\Gamma^*(T)_1$ is already a free edge then we are done. If not, then let $L \subset \Gamma^*(T)$ be a line graph whose free edges are $\Gamma^*(T)_1$ and a free edge of $\Gamma^*(T)$. Using Lemma 3.1 we can find $\alpha \in \mathcal{B}_n$ such that $\Gamma^*(\alpha T)_1$ is a free edge of $\Gamma^*(\alpha T)$.

We now assume that $\Gamma^*(T)$ satisfies Step 1.
Step 2: Find $\alpha \in \mathcal{B}_n$ such that $\Gamma^*(\alpha T)_1 = \Gamma^*(T)_1$ is a free edge of $\Gamma^*(\alpha T)$ and $\Gamma^*(\alpha T)_2$ is adjacent to $\Gamma^*(\alpha T)_1$.

If $\Gamma^*(T)_2$ is already adjacent to $\Gamma^*(T)_1$, then we are done. If not, then let L_1 be a line graph having $\Gamma^*(T)_1$ and $\Gamma^*(T)_2$ as its free edges and let L be the line graph $L_1 \setminus \Gamma^*(T)_1$. An application of Lemma 3.1 shows that there is $\alpha \in \mathcal{B}_n$ which fixes $\Gamma^*(T)_1$ and such that $\Gamma^*(\alpha T)_2$ is adjacent to $\Gamma^*(\alpha T)_1 = \Gamma^*(T)_1$, as required.

We now assume that $\Gamma^*(T)$ satisfies Step 1 and Step 2.
Step 3: This is an inductive step. Let us assume that $\Gamma^*(T)_1$ is a free edge of $\Gamma^*(T)$ and that L' is a line graph with edges $\Gamma^*(T)_1, \Gamma^*(T)_2, \ldots, \Gamma^*(T)_k, k \geq 2$. This step will consist of showing that we can increase k (if $k < n+1$). To simplify notation we may, by a conjugacy, also assume that $T = (t_1, \ldots, t_n)$ where $t_i = (ii+1)$ for $i \leq k$ so that $\Gamma^*(T)_i$ has vertices v_i, v_{i+1}.

Now let L be a line subgraph of $\Gamma^*(T) \setminus L'$ such that one free edge is $\Gamma^*(T)_{k+1}$, while the end vertex at the other end of L is v_{r+1} for some $r \leq k$. Let $\Gamma^*(T)_s$ be the edge of L with end vertex v_{r+1} in it. Then we must have $s \geq k + 1$ and so we can apply Lemma 3.1 to find $\alpha \in \mathcal{B}_n$ such that α fixes L' but $\Gamma^*(\alpha T)_{k+1}$ is now adjacent to v_{r+1}.

If $r = k$, then we are done. So we assume that $r < k$.

We now consider the graph $L' \cup \Gamma^*(\alpha T)_{k+1}$. This graph has edges labeled $1, 2, \dots, r, r+1, r+2, \dots, k-1, k, k+1$. Now if we do $\sigma_{r+2} \dots \sigma_{k-1}\sigma_k$, then the edges will now be labeled $1, 2, \dots, r, r+1, r+3, \dots, k, k+1, r+2$ (respectively). Now we act by σ_{r+1} and we are done since the image of the graph $L' \cup \Gamma^*(\alpha T_{k+1})$ is now a line graph with edges labeled $1, 2, \dots, k+1$.

We illustrate the above in Figure 3, for the case $r = k-3$.

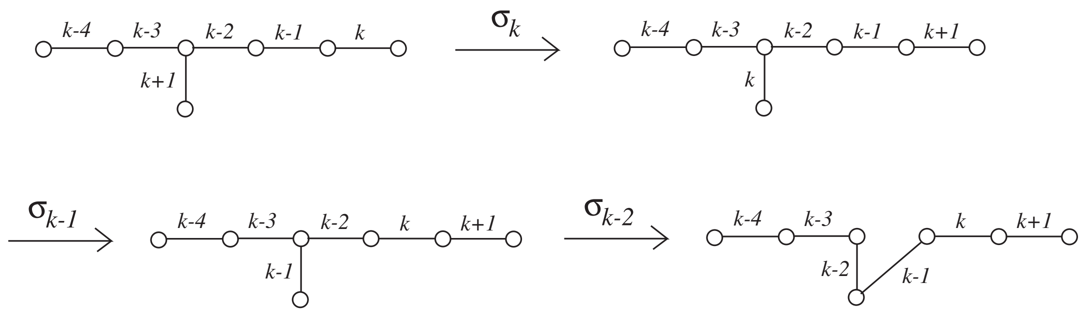

Figure 3.

This completes the proof of Theorem 1.2 for the A_n case.

The next case is $W(B_n)$. Here $W(B_n)$ is the group of all signed permutations of $\pm 1, \pm 2, \dots, \pm n$ and the reflections are in two conjugacy classes. We define $y_1 = 1, y_2 = 2, \dots, y_n = n, y_{n+1} = -1, y_{n+2} = -2, \dots, y_{2n} = -n$, and let

$$\gamma_i = (y_i, y_{n+i}), \quad \beta_{ij} = (y_i, y_j)(y_{n+i}, y_{n+j}), \quad \text{for} \quad 1 \le i \ne j \le n.$$

Any conjugate of γ_i is some γ_j, while if we let $\beta^+_{ij} = \beta_{ij}$ and $\beta^-_{ij} = \gamma_i\beta_{ij}\gamma_i = \gamma_j\beta_{ij}\gamma_j$, then any $W(B_n)$-conjugate of some $\beta^{\pm}_{ij}$ is some $\beta^{\pm}_{rs}$. We also have $\gamma_i\beta_{rs}\gamma_i = \beta_{rs}$ if $i \ne r, s$. The reflections are thus the $\gamma_i, 1 \le i \le n$ and the $\beta^{\pm}_{ij}$ where we may assume that $\le i < j \le n$.

The standard set of generators for $W(B_n)$ is the sequence $(\gamma_1, \beta_{12}, \beta_{23}, \dots, \beta_{n-1n})$. Now let $(\delta_1, \delta_2, \dots, \delta_n)$ be a set of reflection generators so that $\delta_i \in \{\gamma_1, \dots, \gamma_n, \beta^+_{12}, \dots, \beta^+_{n-1n}, \beta^-_{12}, \dots, \beta^-_{n-1n}\}$. Let $\pi = \pi_n : W(B_n) \to S_n$ be the projection so that $\pi(\gamma_i) = id, \pi(\beta_{ij}) = (ij)$. Since $n-1$ is the smallest number of transpositions that S_n can be generated by we see that $n-1$ of the δ_i must be of the form $\beta^{\pm}_{ij}$, with the other being one of the γ_i. Thus we may now assume (Lemma 2.1) that (up to an action of the braid group $\mathcal{B}_n$) we have $\delta_1 = \gamma_i, i \le n$ and $\delta_k \in \{\beta^{\epsilon}_{ij}\}$ for $k \ge 2$ and $\epsilon = \pm$.

Now we must have $< \pi(\delta_2), \dots, \pi(\delta_n) >= S_n$ and so from the $W(A_{n-1}) \cong S_n$ case proved above we see that there is $\alpha \in < \sigma_2, \dots, \sigma_{n-1} > \subset \mathcal{B}_n$ such that

$$\alpha(\delta_1, \dots, \delta_n) = (\gamma_i, \beta^{\epsilon_1}_{u_1u_2}, \beta^{\epsilon_2}_{u_2u_3}, \dots, \beta^{\epsilon_{n-1}}_{u_{n-1}u_n}).$$

Here $u_j \le n$ are distinct and $\epsilon_j = \pm$ for all $j \le n-1$.

We now show how to reduce the number of β^-_{ij}s in $S = (\gamma_i, \beta^{\epsilon_1}_{u_1u_2}, \beta^{\epsilon_2}_{u_2u_3}, \dots, \beta^{\epsilon_{n-1}}_{u_{n-1}u_n})$. Suppose that $r \le n-1$ is the largest such number with $\epsilon_r = -$. Then we conjugate each element of S by γ_{u_r} and we have reduced r. Continue until all the ϵ_k are $+$. Thus we now have $(\gamma_i, \beta_{u_1u_2}, \beta_{u_2u_3}, \dots, \beta_{u_{n-1}u_n})$ and this in turn is conjugate to $(\gamma_i, \beta_{12}, \beta_{23}, \dots, \beta_{n-1n})$ for some i.

Let $C = \beta_{12}\beta_{23} \dots \beta_{n-1n} = (1, n, n-1, \dots, 3, 2)$. Then there is $k \le n$ such that

$$C^k(\gamma_i, \beta_{12}, \beta_{23}, \dots, \beta_{n-1n})C^{-k} = (\gamma_1, C^k\beta_{12}C^{-k}, C^k\beta_{23}C^{-k}, \dots, C^k\beta_{n-1n}C^{-k}).$$

But the effect of the braid $\Pi = (\sigma_2\sigma_3 \dots \sigma_{n-1})^{n-1}$ on all but the first coordinate is to conjugate those coordinates by C. Further this braid fixes the first coordinate. Thus there is some $m \in \mathbb{Z}$ such that $\Pi^m(\gamma_1, C^k\beta_{12}C^{-k}, C^k\beta_{23}C^{-k}, \dots, C^k\beta_{n-1n}C^{-k}) = (\gamma_1, \beta_{12}, \beta_{23}, \dots, \beta_{n-1n})$, as required. This concludes the $W(B_n)$ case.

Next we consider $W(D_n), n \geq 8$. This is the subgroup of index 2 in $W(B_n)$ generated by the $\beta_{ij}^{\pm}$. Here the reflections are the $\beta_{ij}^{\pm}$ and the standard set of generators for $W(D_n)$ is the sequence $(\beta_{12}, \beta_{23}, \dots, \beta_{n-1n}, \beta_{n-1n}^{-})$. We note that the cases $n < 8$ can be checked using MAGMA [MA].

Let $S = (r_1, \dots, r_n)$ be a generating sequence and let $\pi = \pi_n : W(S) \to S_n$ be the epimorphism. Let $S' = \pi(S)$. Then $\Gamma(S')$ is a connected graph with exactly one cycle, which we denote by $C(S')$. Let $q = q(S) \leq n$ be maximal such that $\Gamma(S')_q$ is an edge of $C(S')$. We aim to get $q = n$ by acting on S by an element of $\mathcal{B}_n$. If $q = n$, then we are done. If $q < n$, then we consider two cases (showing in each case how to increase q):
(i) $\Gamma(S')_{q+1}$ is disjoint from $\Gamma(S')_q$; here we act on S (and so on S') by σ_q and note that $q(\sigma_q(S)) = q + 1$.
(ii) If $\Gamma(S')_{q+1}$ is adjacent to $\Gamma(S')_q$, then $\Gamma(S')_{q+1}$ cannot be an edge of $\Gamma(S')$ by the maximality of q. We then check that $q(\sigma_q(S)) > q(S)$.

By the above we may assume that $q(S) = n$. Thus $\Gamma(\pi(r_1, \dots, r_{n-1}))$ is a tree. From the A_{n-1} case considered above we see that there is $\alpha \in \mathcal{B}_{n-1} < \mathcal{B}_n$ such that $\pi(\alpha(r_1, \dots, r_{n-1}))$ is conjugate to $(\beta_{12}, \beta_{23}, \dots, \beta_{n-1n})$; so we may assume that $\pi S = (\beta_{12}, \beta_{23}, \dots, \beta_{n-1n}, \beta_{rs})$ and thus $S = (\beta_{12}^{\epsilon_1}, \beta_{23}^{\epsilon_2}, \dots, \beta_{n-1n}^{\epsilon_{n-1}}, \beta_{rs}^{\epsilon_n})$.

Now the element $(\sigma_1 \dots \sigma_{n-2})^{n-1}$ has the effect of conjugating the first $n-1$ of the elements of S by $\beta_{12}^{\epsilon_1}\beta_{23}^{\epsilon_2} \dots \beta_{n-1n}^{\epsilon_{n-1}}$. Doing this (up to conjugacy) has the effect of acting on the indices r, s by $(1, n, n-1, n-2, \dots, 3, 2)$ and so we may assume that $r < s = n$. Thus we now have $S = (\beta_{12}^{\epsilon_1}, \beta_{23}^{\epsilon_2}, \dots, \beta_{n-1n}^{\epsilon_{n-1}}, \beta_{rn}^{\epsilon_n})$.

Now suppose that $\epsilon_i = -$ and that this choice of i is minimal. Assume that $i < n-1$. Then $\gamma_i\gamma_{i+2} \in W(D_n)$ and if we conjugate by $\gamma_i\gamma_{i+2}$, then we have increased i. Thus we may assume that we have $S = (\beta_{12}, \beta_{23}, \dots, \beta_{n-2,n-1}, \beta_{n-1n}^{\epsilon_{n-1}}, \beta_{rn}^{\epsilon_n})$ and we note that at least one of $\epsilon_{n-1}, \epsilon_n$ is $-$, otherwise S just generates a symmetric group.

We also note that if $r = n-1$, then $\epsilon_{n-1} \neq \epsilon_n$ (else $r_{n-1} = r_n$) and so we have a base.

If $r < n-1$ and $\epsilon_{n-1} = - = \epsilon_n$, then we conjugate by γ_n and see that we have $S = (\beta_{12}, \beta_{23}, \dots, \beta_{n-2,n-1}, \beta_{n-1n}, \beta_{rn})$ which generates a subgroup of S_n i.e. a proper subgroup of D_n. Thus we must have $\epsilon_{n-1} \neq \epsilon_n$.

Now if $r = 1$, then using the action of a power of $(\sigma_1\sigma_2 \dots \sigma_{n-1})^n$ together with conjugation we see that this case is equivalent to the case $r = n-1$. Thus we may now assume that $1 < r \leq n-1$ and that $\epsilon_{n-1} \neq \epsilon_n$.

Next we show that $S_1 = (\beta_{12}, \beta_{23}, \dots, \beta_{n-2,n-1}, \beta_{n-1n}^{-}, \beta_{rn})$ and $S_2 = (\beta_{12}, \beta_{23}, \dots, \beta_{n-2,n-1}, \beta_{n-1n}, \beta_{rn}^{-})$ are equivalent. This is accomplished by first acting on S_2 by $\alpha = \sigma_{n-1}^{-1}\sigma_{n-2}^{-1} \dots \sigma_r^{-1}$ and noting that the result is conjugate to S_1. Thus we may assume that we have S_2 for some value of $1 < r < n$. We let $S(r) = (\beta_{12}, \beta_{23}, \dots, \beta_{n-2,n-1}, \beta_{n-1n}, \beta_{rn}^{-})$.

Lastly we show that each of the $S(r)$ are inequivalent. This will follow from the fact that the products $P(r) = \beta_{12}\beta_{23} \dots \beta_{n-2,n-1}\beta_{n-1n}\beta_{rn}^{-}$ are pair-wise non-conjugate for $1 < r < n$. To see this we note that $P(r)$ is a product of two cycles, one being of length r and containing no -1s, the other of length $n - r$ with some -1s. This is enough to guarantee that the $P(r)$ are pair-wise non-conjugate. There are thus $n-2$ possibilities. This concludes the $W(D_n)$ case.

The remaining cases $(I_3, I_4, F_4, E_6, E_7, E_8)$ are each checked using MAGMA [MA].

We now indicate the reflection matrices that generate the $\mathcal{BC}_n$-classes listed in Theorem 1.2.

For A_n we have

$$M(A_n) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}. \tag{3.1}$$

Then the reflections $s_1, \dots, s_n \in S$ are represented by the matrices $m_1, \dots, m_n$ where m_i is equal to the identity except in the ith row which is equal to the ith row of $M(A_n)$, except that the ii entry is -1.

For type B_n we have

$$M(B_n) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -\sqrt{2} & 2 \end{pmatrix}. \tag{3.2}$$

The reflections in this case are defined as in the A_n case.

For type D_n we have $n-2$ cases indexed by $1 < r < n-1$:

$$M_r(D_n) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & -1 & 2 & -1 & 0 & \dots & 0 & (-1)^{n+r+1} \\ 0 & \dots & 0 & -1 & 2 & -1 & \dots & 0 & (-1)^{n+r+1} \\ 0 & \dots & 0 & 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & (-1)^{n+r+1} & (-1)^{n+r+1} & 0 & \dots & -1 & 2 \end{pmatrix}, \tag{3.3}$$

where the two $(-1)^{n+r+1}$s in the nth column are in rows $r-1$ and r.

One can also easily write down matrices for the cases $I_3, I_4, F_4, E_6, E_7, E_8$ using the $\mathcal{BC}_n$ representatives given in Theorem 1.6.

§4 The n even cases

Suppose in this section that $n = |S| \geq 4$ is even and that (by induction) Theorem 1.1 is true for all $k < n$. We may assume this since the results of §2 (the odd case) show that Theorem 1.1 is true when $|S| = 3$.

Further, we may assume that $n > 9$, since using the results of previous sections we may easily check the cases $n = 4, 6, 8$.

Given a sequence $S = (s_1, \ldots, s_n), s_i = s_{v_i}, \|v_i\| = 1$, of generating reflections we let $M(S)$ denote the matrix $(2v_i.v_j)$ as in §3. Corollary 2.8 shows that the only possible entries of $M(S)$ are $0, \pm 1, \pm\sqrt{2}$.

Lemma 4.1. *Let $W =< S >$ and assume that $\Gamma(S)$ is connected. Then there is $\alpha \in \mathcal{B}_n$ such that if $\alpha(S) = (s'_1, s'_2, \ldots, s'_n)$, then $\Gamma((s'_1, s'_2, \ldots, s'_{n-1}))$ is connected.*

Proof. Suppose that $\Gamma((s_1, s_2, \ldots, s_{n-1}))$ is not connected (if it is then we are done). Let $I_1 \cup \cdots \cup I_r$ be the connected components of $\Gamma((s_1, s_2, \ldots, s_{n-1}))$. Then by Lemma 2.1 there is $\alpha_1 \in \mathcal{B}_{n-1}$ such that all of the elements of I_1 are in positions $1, 2, \ldots, |I_1|$ of $\alpha_1(S)$. Now since I_1 and $I_t, t \neq 1$ are disjoint in $\Gamma(S)$ we see that $ab = ba$ for $a \in I_1, b \in I_t$. Thus $I_1, I_2, \ldots, I_r$ are still the components of $\alpha_1(S) - \{s_n\}$. We may thus repeat to get $\alpha_2 \in \mathcal{B}_{n-1}$ such that all of the elements of I_1 are in positions $1, 2, \ldots, |I_1|$ of $\alpha_2\alpha_1(S)$ and all of the elements of I_2 are in positions $|I_1| + 1, \ldots, |I_1| + |I_2|$ of $\alpha_2\alpha_1(S)$. Continuing we see that there is $\beta \in \mathcal{B}_{n-1}$ such that the above is true and that the last $|I_r|$ elements of $\beta(s_1, \ldots, s_{n-1})$ are those in I_r. We may also assume that $s_{n-1} \in I_r$. Thus replacing S by $\beta(S)$ and by induction we may assume that each component of $\Gamma((s_1, s_2, \ldots, s_{n-1}))$ is in standard form in the positions indicated above. In particular (see Lemma 2.1 (iv)) we may assume that $\Gamma((I_r \setminus \{s_{n-1}\}))$ is connected. Thus if $\Gamma(S)_n$ and $\Gamma(S)_{n-1}$ are disjoint, then acting by σ_{n-1} gives the result since the action of σ_{n-1} just interchanges the labels on these edges.

If $\Gamma(S)_n$ and $\Gamma(S)_{n-1}$ are not disjoint, then acting by σ_{n-1}^2 replaces $S = (\ldots, s_{n-1}, s_n)$ by $(\ldots, s_n s_{n-1} s_n, s_n s_{n-1} s_n s_{n-1} s_n)$ and we note that $\Gamma(\sigma_{n-1}^2(S) \setminus \{s_n s_{n-1} s_n s_{n-1} s_n\})$ is now connected since $s_n s_{n-1} s_n$ does not commute with some element of each of the $I_1, \ldots, I_{r-1}$ and does not commute with $s_n s_{n-1} s_n s_{n-1} s_n$ either. $\square$

Given $S = (s_1, \ldots, s_n)$ we let $S(1, 2, \ldots, n-1) = (s_1, \ldots, s_{n-1})$. By Lemma 4.1 we may assume that $\Gamma(S(1, 2, \ldots, n-1))$ is connected. Since $|S(1, 2, \ldots, n-1)| = n - 1$ is odd we see from §2 that $< S(1, 2, \ldots, n-1) >$ is an irreducible finite reflection group and that there is $\alpha \in \mathcal{B}_{n-1} < \mathcal{B}_n$ such that $\alpha(S(1, 2, \ldots, n-1))$ is in one of the canonical forms determined in Theorem 1.2 and §3. We now consider each case separately. Before doing so we will need:

Lemma 4.2. *Suppose that $W =< S >, S = (s_1, s_2, \ldots, s_n)$ has finite $\mathcal{B}_n$-orbit. Then*
(i) for any $S' = \{t_1, \ldots, t_r\} \subset \{s_1, s_2, \ldots, s_n\}$ we have $det(t_i.t_j) \neq 0$.
(ii) If $\Gamma(S)$ is connected and $P = s_i s_{i+1} \ldots s_{i+k}, k \geq 1$, does not have finite order, then the $\mathcal{B}_n$-orbit of S is infinite.

Proof. (i) follows since the t_i are linearly independent. (ii) follows from Proposition 2.3 applied to $< \sigma_i, \sigma_{i+1}, \ldots, \sigma_{i+k-1} >$. $\square$

Given a sequence $S = (s_1, \ldots, s_n), s_i = s_{v_i}, \|v_i\| = 1$ of generating reflections we let $M(S)$ denote the matrix $(v_i.v_j)$ as in §3. By Proposition 2.7 the entries of $M(S)$ are in the set

$\{0, \pm 1, \pm\sqrt{2}\}$. We let $\gamma(S)$ denote the labeled graph with vertices $v_1, \dots, v_n$ and with an edge between v_i and v_j if $v_i.v_j \neq 0$ and with the label being $v_i.v_j$.

THE $\mathbf{A_n}$ CASE :

First assume that $\Gamma(S(1, \dots, n-1))$ is of type A_{n-1} so that by Theorem 1.2 we may assume that $\Gamma(S(1, \dots, n-1))$ is a line graph with $n-1$ edges. Thus the homomorphism $W(S(1, \dots, n-1)) \to S_n$ induced by sending s_i to the transposition $(i, i+1)$ is an isomorphism. The matrix (3.1) shows what $M(S(1, \dots, n-1))$ looks like.

We will first determine what happens if we have at least one label equal to $\pm\sqrt{2}$.

Lemma 4.3. *Assume the above set-up and that $\pm\sqrt{2}$ occurs in the last column of $M(S)$.*

(i) If there are $1 \leq t \neq r \leq n-1$ such that $M(S)_{tn} = \pm\sqrt{2}, M(S)_{rn} = \pm 1$, then the $\mathcal{B}_n$-orbit of S is infinite.

(ii) If there is $1 < t < n-1$ such that $M(S)_{tn} = \pm\sqrt{2}, M(S)_{t-1,n} = M(S)_{t+1,n} = 0$, then the $\mathcal{B}_n$-orbit of S is infinite.

(iii) If there are $1 \leq t < u-1 < n-1$ such that $M(S)_{tn} = \pm\sqrt{2}, M(S)_{u,n} = \pm\sqrt{2}$, then the $\mathcal{B}_n$-orbit of S is infinite.

(iv) If none of (i), (ii) (iii) occur, then there is $1 \leq t < r \leq n-1$ such that $M(S)_{un} = 0$ for $u = 1, \dots, t-1$; $M(S)_{un} = \pm\sqrt{2}$ for $u = t, \dots, r$ and $M(S)_{un} = 0$ for $u = r+1, \dots, n-1$. If $|t-r| > 2$ or if $t = r \neq 1, n-1$, then the $\mathcal{B}_n$-orbit of S is infinite. In all other cases $< S >= W(B_n)$.

Proof. (i) Assume that $|t-r|$ is minimal among all such choices. Assume $t < r$ (the other case is similar). Thus we must have $M(S)_{un} = 0$ for all $t < u < r$. Further, by conjugating by $diag(1, 1, \dots, 1, \pm 1)$ we may assume that $M(S)_{tn} = \sqrt{2}$. Thus if $r - t > 1$, then we may act by σ_{r-1}^{-1} and reduce $r-t$. Thus we assume $r = t+1$. Now by Lemma 2.1 we can move s_t, s_{t+1} into the $n-2, n-1$ positions. Thus we now have

$$M((s_{n-2}, s_{n-1}, s_n)) = \begin{pmatrix} 2 & -1 & \sqrt{2} \\ -1 & 2 & \pm 1 \\ \sqrt{2} & \pm 1 & 2 \end{pmatrix},$$

so that

$$s_{n-2}s_{n-1}s_n = \begin{pmatrix} 2 \mp \sqrt{2} & \pm\sqrt{2} & \pm 1 - \sqrt{2} \\ \pm\sqrt{2} - 1 & 0 & \mp 1 \\ \sqrt{2} & \pm 1 & -1 \end{pmatrix}.$$

One checks that this matrix has an eigenvalue greater than 1; thus we see that the $< \sigma_{n-2}, \sigma_{n-1} >$-orbit of (s_{n-2}, s_{n-1}, s_n) is infinite (Lemma 4.2).

(ii) In this situation Lemma 2.1 allows us to assume that $t = n-2$. Again we may assume that $M(S)_{tn} = \sqrt{2}$. Here $P = s_{n-3}s_{n-2}s_{n-1}s_n$ has infinite order ($P^2 - I_4$ has rank 1 and all eigenvalues of P^2 are 1) and so Lemma 4.2 gives the result.

(iii) In view of (i) and (ii) above we may assume that $M(S)_{tn} = \sqrt{2}$ and that $M(S)_{yn} = 0$ for $t < y < u$. As in (ii), if $u - t > 2$, then acting by σ_{u-1} reduces $u-t$. Thus we may assume that $u = t+2$ and then that $u = n-1$ (Lemma 2.1). One calculates the products $P = s_{n-3}s_{n-2}s_{n-1}s_n$ in this situation and shows that they are infinite.

(iv) The first statement follows easily since we are assuming that $\pm\sqrt{2}$ occurs in the last column of $M(S)$. We may now conjugate so as to have $M(S)_{tn} = \sqrt{2}$. If $M(S)_{t+1,n} = -\sqrt{2}$, then one shows that the $\mathcal{B}_n$-orbit is infinite by looking only at s_t, s_{t+1}, s_n (which one moves into the $n-2, n-1, n$ positions). It follows similarly that we can now assume $M(S)_{u,n} = \sqrt{2}$ for all $t \leq u \leq r$.

Now one checks that if $M(S)_{u,n} = \sqrt{2}$ for all $u = t, t+1, t+2$, then we have an infinite $< \sigma_{n-3}, \sigma_{n-2}, \sigma_{n-1} >$-orbit (after moving s_t, s_{t+1}, s_{t+2} into the $n-3, n-2, n-1$ positions). Thus we now may assume that $r = t+1$.

Now let $\alpha = (\sigma_1\sigma_2 \dots \sigma_t)(\sigma_1\sigma_2 \dots \sigma_{t-1}) \dots (\sigma_1\sigma_2)(\sigma_1)$. Then $\alpha(S)$ has matrix

$$\begin{pmatrix}
2 & -1 & -1 & -1 & \dots & -1 & 0 & \dots & 0 & -\sqrt{2} \\
-1 & 2 & -1 & -1 & \dots & -1 & 0 & \dots & 0 & 0 \\
-1 & -1 & 2 & -1 & \dots & -1 & 0 & \dots & 0 & 0 \\
-1 & -1 & -1 & 2 & \dots & -1 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots \\
-1 & -1 & -1 & -1 & \dots & 2 & -1 & \dots & \vdots & 0 \\
0 & 0 & 0 & 0 & \dots & -1 & 2 & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & -1 & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & -1 & 2 & 0 \\
-\sqrt{2} & 0 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & 2
\end{pmatrix} \tag{4.1}$$

where the last -1 in the first row is in column $t+2$. Now let $\beta = \sigma_2\sigma_3 \dots \sigma_t$. Then the matrix of $\beta\alpha(S))$ is

$$\begin{pmatrix}
2 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & -\sqrt{2} \\
0 & 2 & -1 & -1 & \dots & -1 & 0 & \dots & 0 & 0 \\
0 & -1 & 2 & -1 & \dots & -1 & 0 & \dots & 0 & 0 \\
0 & -1 & -1 & 2 & \dots & -1 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots \\
0 & -1 & -1 & -1 & \dots & 2 & -1 & \dots & \vdots & 0 \\
0 & 0 & 0 & 0 & \dots & -1 & 2 & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & -1 & 0 \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & -1 & 2 & 0 \\
-\sqrt{2} & 0 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & 2
\end{pmatrix}$$

where the last -1 in the second row is in column $t+3$. Now let $\gamma = \sigma_k^{-1} \dots \sigma_3^{-1}\sigma_2^{-1}$. Then the matrix of $\gamma\beta\alpha(S)$ is as in (4.1) except that now the last -1 in the first row is in column $t+1$. Repeating these last steps shows that these reflections generate $W(B_n)$. This concludes the proof of Lemma 4.3 and so of all cases where $\pm\sqrt{2}$ occurs. $\square$

Now assume that only $0, \pm 1$s occur in $M(S)$.

Let L_m denote the line graph with m vertices in the order $1, 2, \dots, m$ and K_m the complete graph. For a graph Γ with m vertices let $M(\Gamma)$ denote the $m \times m$ matrix whose ij entry is -1 if ij is an edge of Γ and is otherwise 0. If Γ_1, Γ_2 are graphs with vertices $v_1, \dots, v_m$ and $w_1, \dots, w_t$, then by $\Gamma_1 * \Gamma_2$ we mean the graph with vertices $v_1, \dots, v_m, w_1, \dots, w_t$, with edges those of Γ_1, Γ_2 together with an edge between v_m and w_1. Thus $L_{n+m} \cong L_n * L_m$, for example. For $r \geq s \geq 1$ let

$$\sigma_{rs} = \sigma_r\sigma_{r-1} \dots \sigma_s; \quad \Sigma_{rs} = \sigma_{rs}\sigma_{r,s-1}\sigma_{r,s-2} \dots \sigma_{rr}.$$

Now we may describe the present situation by giving the vector $V(S) = (v_i.v_n), S = (r_{v_1}, \dots, r_{v_n})$, of length $n-1$ since the matrix $M(s_1, \dots, s_{n-1})$ is $M(A_{n-1})$. Here the entries of $V(S)$ are $0, -1, 1$. We will say that *V(S) is finite* if the corresponding group generated by reflections is finite (otherwise $V(S)$ is infinite). Note that by Lemma 2.1 if a subsequence of $V(S)$ is infinite, then $V(S)$ is infinite. We also note that $V(S) \neq (0, \dots, 0)$ as we are assuming that $< S >$ is irreducible. By conjugating by $diag(1, 1, \dots, 1, \pm 1)$ (if necessary) we may assume that the first non-zero entry of $V(S)$ is -1. We define $\lambda(S)$ to be the number of non-zero entries in $V(S)$. We define an equivalence relation on $1, \dots, n-1$; this is generated by having i and $i+1$ related if $v_i.v_n$ and $v_{i+1}.v_n$ are both non-zero. We will call the equivalence classes *components.* There is a dual equivalence relation where i and $i+1$ are related if $v_i.v_n$ and $v_{i+1}.v_n$ are both zero. We will call these equivalence classes *dual components.* We let $c(S)$ denote the number of components.

Lemma 4.4.
(i) For $n \geq 8$, $V(S) = (-1, \dots, -1)$ is infinite.
(ii) If there are $1 \leq r < s \leq n-1$ and $\epsilon = \pm 1$ with $s-r$ even, $v_r.v_n = \epsilon, v_s.v_n = \epsilon$ and $v_i.v_n = 0$ for $r < i < s$, then $V(S)$ is infinite.
(iii) If $v_i.v_n = 0, n > i > 1$, and $\sigma_i(S) = (w_1, \dots, w_n)$ then $V(w_1, \dots w_i, w_{i+2}, \dots, w_n) = (v_1.v_n, v_2.v_n, \dots, v_i.v_n, -v_{i+2}.v_n, \dots, -v_{n-1}.v_n)$.
(iv) If there are $1 \leq r < s \leq n-1$ and $\epsilon = \pm 1$ with $s-r$ odd, $v_r.v_n = \epsilon, v_s.v_n = -\epsilon$ and $v_i.v_n = 0$ for $r < i < s$, then $V(S)$ is infinite.
v) If $\lambda(S) \geq 8$, then $V(S)$ is infinite.
(vi) Let $\sigma = \sigma_{n-2}\sigma_{n-3}\dots\sigma_1$. Then

$$V(\sigma(S)) = (v_1.v_n - v_2.v_n + v_3.v_n - \dots + (-1)^n v_{n-1}.v_n, v_1.v_n, v_2.v_n, v_3.v_n, \dots, v_{n-2}.v_n).$$

Proof. (i) One checks that for $n = 8$ the product of all the reflections has infinite order and is not a diagonal matrix.

(ii) One checks that $(1, 1, \dots, 1)$ is in the nullspace of $M(v_r, \dots, v_s)$, contradicting Lemma 4.2 (i).

(iii) One checks this using the action given in §1.

(iv) Using (iii) one can reduce this case to (ii), from which the result follows.

(v) If $\lambda(S) \geq 8$, then using (iii) we can reduce to $V(S) = (\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$. If two of these signs are different, then we can assume that they are adjacent and the result follows from (iv). If they are all the same, then the result follows from (i).

(vi) One inducts. $\square$

From (v) above we now need only check the cases $0 < \lambda(S) \leq 7$. Further, by (vi) we may assume that $v_1.v_n = 0$. We will use the notation 1^6 for the sequence $1, 1, 1, 1, 1, 1$ etc.

Lemma 4.5. *The following cases are all infinite:*

(*i*) $(0, 1^7)$.

(*ii*) $(0, 1^6, 0), (0, 1^5, 0, -1), (0, 1^4, 0, -1^2),$
$(0, 1^3, 0, -1^3), (0, 1^2, 0, -1^4), (0, 1, 0, -1^5), (0, 0, -1^6)$.

(*iii*) $(0, 1^5, 0)$.

(*iv*) $(0, 1^4, 0, -1, 0), (0, 1^4, 0, 0, 1), (0, 0, 1^4, 0, -1), (0, 1, 0, -1^4, 0), (0, 1, 0, 0, 1^4), (0, 0, 1, 0, 1^4)$.

(*v*) $(0, 1^2, 0, -1^3, 0), (0, 1^2, 0, 0, 1^3), (0, 0, 1^2, 0, -1^3), (0, 1^3, 0, -1^2)$.

(*vi*) $(0, 1^3, 0, -1, 0, 1), (0, 1, 0, -1^3, 0, 1), (0, 1, 0, -1, 0, 1^3)$.

(*vii*) $(0, 1^2, 0, -1^2, 0, 1), (0, 1^2, 0, -1, 0, 1^2), (0, 1, 0, -1^2, 0, 1^2)$.

(*viii*) $(0, 1^2, 0, -1, 0, 1, 0, -1), (0, 1, 0, -1^2, 0, 1, 0, -1),$
$(0, 1, 0, -1, 0, 1^2, 0, -1), (0, 1, 0, -1, 0, 1, 0, -1^2)$.

(*ix*) $(0, 1, 0, -1, 0, 1, 0, -1, 0, 1)$.

(*x*) $(0, 1^3, 0, -1, 0, 0), (0, 1^3, 0, 0, 1, 0), (0, 1, 0, -1^3, 0, 0), (0, 1, 0, 0, 1^3, 0),$
$(0, 0, 1, 0, -1^3, 0), (0, 0, 1, 0, 0, 1^3), (0, 0, 1^3, 0, -1, 0), (0, 0, 1^3, 0, 0, 1)$.

(*xi*) $(0, 1^2, 0, -1, 0, 1, 0), (0, 1^2, 0, -1, 0, 0, -1), (0, 1^2, 0, 0, 1, 0, -1), (0, 1, 0, -1^2, 0, 1),$
$(0, 1, 0, -1, 0, 1^2), (0, 1, 0, 0, 1, 0, 1^2), (0, 1, 0, 0, 1, 0, -1, -1)$.

(*xii*) $(0, 1, 0, -1, 0, 1, 0, -1)$.

(*xiii*) $(0, 1^3, 0^4), (0, 0, 1^3, 0, 0), (0^4, 1^3, 0)$.

(*xiv*) $(0, 1, 0, -1, 0, 1, 0), (0, 1, 0, 0, 1, 0, -1), (0, 1, 0, -1, 0, 0, 0, 1)$
$(0, 0, 1, 0, -1, 0, 0, 1), (0, 0, 1, 0, 0, 1, 0, -1), (0, 0, 0, 1, 0, -1, 0, 1)$.

Proof. One checks in each case that the corresponding product of reflections has infinite order and is not a scalar matrix. □

Lemma 4.5 (i) together with Lemma 4.4 (iii) shows that all the $\lambda(S) = 7$ cases are infinite.

Assume that $\lambda(S) = 6$. Here we use Lemma 4.5 (ii), noting that each case with $\lambda(S) = 6$ can now be reduced to one of these cases. This does $\lambda(S) = 6$.

Now assume that $\lambda(S) = 5$. Then the $c(S)$ equivalence classes determine a partition of 5. If $c(S) = 1$, then Lemma 4.5 (iii) does this case.

If $c(S) = 2$ and the partition is $5 = 4 + 1$, then Lemma 4.5 (iv) does this case. If $c(S) = 2$ and the partition is $5 = 3 + 2$, then Lemma 4.5 (v) does this case. This does the $c(S) = 2$ cases.

If $c(S) = 3$ and the partition is $5 = 3 + 1 + 1$, then Lemma 4.5 (vi) does this case. If $c(S) = 3$ and the partition is $5 = 2 + 2 + 1$, then Lemma 4.5 (vii) does this case. This does the $c(S) = 3$ cases.

If $c(S) = 4$ and the partition is $5 = 2 + 1 + 1 + 1$, then Lemma 4.5 (viii) does this case.

If $c(S) = 5$ and the partition is $5 = 1 + 1 + 1 + 1 + 1$, then Lemma 4.5 (ix) does this case.

This concludes all of the $\lambda(S) = 5$ cases.

Now assume that $\lambda(S) = 4$. Lemma 4.5 (x) does the partition $4 = 3 + 1$. Lemma 4.5 (xi) does the partition $4 = 2 + 1 + 1$. Lemma 4.5 (xii) does the partition $4 = 1 + 1 + 1 + 1$.

We are thus left (in the case $\lambda(S) = 4$) with the partitions $4 = 4$ and $4 = 2+2$; we deal with these simultaneously in the following way. Let σ be as in Lemma 4.4 (vi). Then by that same result we see that there is $k \in \mathbb{Z}$ such that $V(\sigma^k(S)) = (0, \ldots, 0, \pm 1, \pm 1, 0, \ldots, 0, -1, -1, 0)$ (in the $4 = 4$ partition case there are no 0s between $\epsilon(= -1)$ and -1); thus $V(\sigma^{k+1}(S)) = (0, \ldots, 0, \pm 1, \pm 1, 0, \ldots, 0, -1, -1)$ and so, $V(\sigma^{k+2}(S)) = (0, \ldots, 0, \pm 1, \pm 1, 0, \ldots, 0, -1)$. So the following result gives what we need:

Lemma 4.6. *Let $\epsilon = \pm 1$ and let $V(S) = (0, \ldots, 0, \epsilon, \epsilon, 0, \ldots, 0, -1)$, where the first two non-zero entries are in positions $u, u + 1 \leq n - 2$. Then*
(i) $< S >$ is infinite if $\epsilon = -1$ and $n - u$ is even, or $\epsilon = 1$ and $n - u$ is odd.
(ii) $< S >$ is finite otherwise.

Proof. (i) follows from Lemma 4.4. (ii) follows from the standard form (3.3) in the D_n case if $u < n - 3$; while if $u = n - 3$, then $M(\sigma_{n-1}(S))$ is the standard form for $M(A_n)$. □

This completes the cases where $\lambda(S) = 4$.

Assume that $\lambda(S) = 3$. The partition $3 = 2 + 1$ is essentially the content of Lemma 4.6 and Lemma 4.5 (xiv) does the partition $3 = 1 + 1 + 1$. If $c(S) = 1$, then Lemma 4.5 (xiii) does this case except for $(0^{n-3}, 1^3)$ which is covered by Lemma 4.6.

Lemma 4.7. *Assume that $\lambda(S) = 2$ and that the non-zero entries of $V(S)$ are in positions $1 \leq r < s \leq n - 1$. Then either S is $\mathcal{BC}_n$-equivalent to S' with $\lambda(S') = 1$ or $< S >$ is infinite.*

Proof. We first note that $(0^2, 1, 0^2, 1, 0^2)$ is infinite and so this reduces us to considering the cases (a) $s = r + 1, r + 2$; (b) $r = 1, 2$; (c) $s = n - 2, n - 1$.

If (a), then Lemma 4.4 (vi) shows acting by σ^k for some $k \in \mathbb{Z}$ will reduce $\lambda(S)$ to 1. This also applies if $r = 1, s = n, n - 1$ or $r = 2, s = n$. For (b) (but not (a)), the following infinite cases (which are easily checked) give the result:

$$(1, 0^2, 1, 0^3), (1, 0^3, -1, 0^2), (1, 0^6, 1, 0),$$
$$(0, 1, 0^2, 1, 0^3), (0, 1, 0^3, -1, 0), (0, 1, 0^5, -1).$$

For (c) we use the transposes of (b). □

Lastly, if $\lambda(S) = 1$ and the non-zero entry is in entry r, then (since $n \geq 10$) $< S >$ is finite if and only if $r = 1, 2, n - 2, n - 1$.

This concludes the A_n case.

THE $\mathbf{B_n}$ CASE :

So assume that $\Gamma(S(1, \ldots, n - 1))$ is of type B_n. See (3.2) for the standard form of $M(B_{n-1})$. Again we consider the vector $V(S)$, which has entries $0, \pm 1, \pm\sqrt{2}$. The presence of the $\sqrt{2}$ in $M(S_{n-1})$ now prohibits many more cases. First we note that we have the following equivalent of Lemma 4.4:

Lemma 4.8.
(i) If there are $1 \leq r < s \leq n - 2$ and $\epsilon = \pm 1$ with $s - r$ even, $v_r.v_n = \epsilon, v_s.v_n = \epsilon$ and $v_i.v_n = 0$ for $r < i < s$, then $V(S)$ is infinite.
(ii) If $v_i.v_n = 0, n - 1 > i > 1$, and $\sigma_i(S) = (w_1, \ldots, w_n)$ then $V(w_1, \ldots w_i, w_{i+2}, \ldots, w_n) = (v_1.v_n, v_2.v_n, \ldots, v_i.v_n, -v_{i+2}.v_n, \ldots, -v_{n-1}.v_n)$.

(iii) If there are $1 \leq r < s \leq n-2$ and $\epsilon = \pm 1$ with $s-r$ odd, $v_r.v_n = \epsilon, v_s.v_n = -\epsilon$ and $v_i.v_n = 0$ for $r < i < s$, then $V(S)$ is infinite.
(iv) Let $\sigma = \sigma_{n-2}\sigma_{n-3}\ldots\sigma_1$. Then

$$V(\sigma(S)) = (v_1.v_n - v_2.v_n + v_3.v_n - \cdots + (-1)^n\sqrt{2}v_{n-1}.v_n,\\ v_1.v_n, v_2.v_n, v_3.v_n, \ldots, v_{n-3}.v_n, \sqrt{2}v_{n-2}.v_n - v_{n-1}.v_n).$$

Proof. This is similar to the proof of Lemma 4.4. □

First we will show that $\pm\sqrt{2}$ does not occur in the entries of $V(S)$. First assume that $V(S) = (\ldots, \sqrt{2})$. One checks that

$$(\ldots, 1, \sqrt{2}), (\ldots, -1, -1, \sqrt{2}), (\ldots, \pm\sqrt{2}, \sqrt{2}), (\ldots, \pm\sqrt{2}, -1, \sqrt{2})$$

are all infinite and so the only cases that could possibly be finite are $(0, 0, \ldots, 0, -1, \sqrt{2})$. If one uses the σ as in Lemma 4.8 (iv) this case becomes $M(B_n)$ and so we are done.

If $\pm\sqrt{2}$ occurs in a position other than the last, then we can use Lemma 4.8 (iv) to put the $\sqrt{2}$ into the $n-2$ position of $V(S)$. We then note that

$$(\ldots, \sqrt{2}, 0), (\ldots, \sqrt{2}), (\ldots, 1, \sqrt{2}, -1), (\ldots, -1, \sqrt{2}, 1), (\ldots, 0, 0, \sqrt{2}, -1)$$

are all infinite. Thus all the cases where $\pm\sqrt{2}$ occurs in $V(S)$ are infinite.

We may now assume that the only entries in $V(S)$ are $0, \pm 1$. Noting that

$$(\ldots, 0, 1, 0), (\ldots, 0, \pm 1, 1), (\ldots, \pm 1, \pm 1, \pm 1), (\ldots, 0^3, 1)$$

are all infinite leaves us with the case $(1, 0, \ldots, 0)$, which clearly gives $W(B_n)$. This concludes the B_n case.

THE $\mathbf{D_n}$ CASE :

So assume that $\Gamma(S(1, \ldots, n-1))$ is of type D_n. Again we consider the vector $V(S)$, which has entries $0, \pm 1, \pm\sqrt{2}$. Here we may assume that $M(S(1, \ldots, n-1))$ has type $M_r(D_{n-1})$ as given in (3.3). The analogue of Lemma 4.4 in this case is:

Lemma 4.9.
(i) If there are $1 \leq r < s \leq n-2$ and $\epsilon = \pm 1$ with $s-r$ even, $v_r.v_n = \epsilon, v_s.v_n = \epsilon$ and $v_i.v_n = 0$ for $r < i < s$, then $V(S)$ is infinite.
(ii) If $v_i.v_n = 0, n-1 > i > 1$, and $\sigma_i(S) = (w_1, \ldots, w_n)$ then $V(w_1, \ldots w_i, w_{i+2}, \ldots, w_n) = (v_1.v_n, v_2.v_n, \ldots, v_i.v_n, -v_{i+2}.v_n, \ldots, -v_{n-1}.v_n)$.
(iii) If there are $1 \leq r < s \leq n-2$ and $\epsilon = \pm 1$ with $s-r$ odd, $v_r.v_n = \epsilon, v_s.v_n = -\epsilon$ and $v_i.v_n = 0$ for $r < i < s$, then $V(S)$ is infinite.
(iv) Let $\sigma = \sigma_{n-2}\sigma_{n-3}\ldots\sigma_1$. Then

$$\begin{aligned} V(\sigma(S)) =& (v_1.v_n - v_2.v_n + v_3.v_n - \cdots + (-1)^{r+n}v_{n-1}.v_n, v_1.v_n, v_2.v_n, v_3.v_n, \ldots, v_{r-2}.v_n, \\ &, v_{r-1}.v_n + (-1)^{r+n}v_{n-1}.v_n, v_r.v_n + (-1)^{r+n}v_{n-1}.v_n, v_{r+1}.v_n, \ldots \\ &, v_{r-1}.v_n - (-1)^{r+n}(v_{n-2}.v_n - v_{n-1}.v_n)). \end{aligned}$$

Proof. This is similar to the proof of Lemma 4.4. □

We first consider the case where $\pm\sqrt{2}$ occurs in $V(S)$. By applying Lemma 4.3 to $(s_1, s_3, \dots s_{n-2}, s_n)$ we see that either a) $V(S) = (\sqrt{2}, 0, \dots, 0, *)$; or b) $(0, \dots, 0, \sqrt{2}, *)$ (where $* \in \{0, \pm 1, \pm\sqrt{2}\}$); or c) there is $1 < t < n-1$ such that $V(S) = (0, \dots, 0, \sqrt{2}, \sqrt{2}, 0, \dots, 0, *)$, with the first $\sqrt{2}$ in the $t-1$ position.

For a) we note that $(\sqrt{2}, 0^4, *)$ is infinite for $r = 2, 3$ and for all $* \in \{0, \pm 1, \pm\sqrt{2}\}$. This does case a) and b) is similar.

For c) we note that $(0^a, \sqrt{2}, \sqrt{2}, 0^b, *)$ is infinite for all $a, b \geq 0, a + b = 4$, and for all $1 < r < 6$. This does c).

We now assume that the entries of $V(S)$ are in $\{0, \pm 1\}$.

We may now apply Lemma 4.6 to $(s_1, s_2, \dots, s_{n-2}, s_n)$, since $(s_1, s_2, \dots, s_{n-2})$ is in standard form for type A_{n-2}. We deduce that $V(S)$ is of the form a) $(0, \dots, 0, \epsilon, \epsilon, 0, \dots, 0, 1, *)$; b) $(0, \dots, 0, 1, 1, *)$; or c) $(0, \dots, 0, 1, 0, \dots, 0, *)$.

From the fact that $(0^a, 1^2, 0^b, \pm 1, *)$ is infinite for all $a, b \geq 0, a + b = 0$ we see that all a) cases are infinite. For b) we have that $(0^5, 1^2, *)$ is infinite except in the case $(0^5, 1^2, 0)$. For c) one notes that $(1, 0, \dots, 0)$ and $(0, \dots 0, , 1, 1)$ are finite and that all other cases are infinite.

We now note that $(0, \dots, 1, 1, 0), (0, \dots, 0, 1, 1)$ are both in the $\mathcal{B}_n$-orbit of $(1, 0, \dots, 0)$ (use Lemma 4.9 (iv)), and that $(1, 0, \dots, 0)$ gives a base for the D_n case.

This completes the proof of Theorem 1.1. □

§5 The A_n representation of $\mathcal{B}_n$

Fix $n \geq 3$ and let $PO_n(A_n)$ denote the set of conjugacy classes as in §1. Each $[S] \in PO(A_n)$ can be represented by a labeled graph $\Gamma([S])$ as follows: Let $\Gamma'(S)$ be the graph with vertices $1, 2, \dots, n+1$ and an edge $\{i, j\}$ labeled k if (ij) is the kth entry of S. We get $\Gamma([S])$ by forgetting the labels on the vertices (or equivalently by taking the S_{n+1}-orbit of $\Gamma'(S)$). It is clear that $PO(A_n)$ is in bijective correspondence with the set of edge-labeled trees with n edges. In this section it will be convenient to redefine the $\mathcal{B}_n$ action: $\sigma_i(s_1, s_2, \dots, s_n) = (s_1, \dots, s_{i-1}, s_i s_{i+1} s_i^{-1}, s_i, s_{i+2}, \dots, s_n)$; $\sigma_i^{-1}(s_1, s_2, \dots, s_n) = (s_1, \dots, s_{i-1}, s_{i+1}, s_{i+1}^{-1} s_i s_{i+1}, s_{i+2}, \dots, s_n)$. This works since $\sigma_i \mapsto \sigma_i^{-1}$ extends to an isomorphism of $\mathcal{B}_n$.

We have shown in Theorem 1.2 that $\mathcal{B}_n$ acts transitively on $PO(A_n)$. Now let $\Gamma_0 = \Gamma([(12), (23), \dots, (n, n+1)])$ and define $\delta_s = \sigma_1\sigma_2 \dots \sigma_{s-1}, \tau_s = \delta_s^{s+1}, 1 < s \leq n$ as in [CW], except that their action was on the right and ours is on the left. Let $H_n = < \tau_2, \tau_3, \dots, \tau_n >$. Then as in [CW] we note that H_n stabilises Γ_0.

We also let $u_{rs} = (\sigma_r\sigma_{r+1} \dots \sigma_{s-1})^{s-r+2}$ for $1 \leq r < s < n$ so that $u_{1s} = \tau_s$. Then $< u_{rs} >= H_n$. We will also let

$$H_{n3} = < u_{i,j} | 1 \leq i < j \leq n, j - i \leq 2 > .$$

If Γ is an edge-labeled graph and Q is a subset of the edge labels, then we let Γ_Q denote the full labeled subgraph of Γ having only the edges with labels in Q. We next prove that H_n acts transitively on the complement of Γ_0; in fact we prove:

Proposition 5.1. *The group $H_{n3}, n \geq 3$, acts transitively on $PO(A_n) \setminus \{\Gamma_0\}$.*

Proof. Let $St_n = ((1, n+1), (2, n+1), \dots, (n, n+1))$, so that $\Gamma(St_n)$ is the star graph with n edges. Proposition 5.1 will follow directly from:

Lemma 5.2. *Each graph in $PO(A_n) \setminus \{\Gamma_0\}$, is equivalent under H_{n3} to the star graph.*

Proof. We prove Lemma 5.2 by induction on $n \geq 3$, the case $n = 3$ being easy to check.

Lemma 5.3. *(i) For $1 \leq i < n$ the action of σ_i on $PO(A_n)$ is a permutation with $2-$cycles and 3-cycles, the 2-cycles are when $\Gamma_{\{i,i+1\}}$ has two components. The element $u_{i,i+1} = \sigma_i^3$ has order 2 and $u_{i,i+1}(\Gamma) = (i, i+1)(\Gamma)$ if $\Gamma_{\{i,i+1\}}$ has two components.*
(ii) The element $u_{i,i+2}$ acts on $PO(A_n)$ so as to have three and nine cycles. The nine-cycles are when $\Gamma_{\{i,i+1,i+2\}}$ has two components. Also, $u_{i,i+2}$ fixes Γ if and only if $\Gamma_{\{i,i+1,i+2\}}$ is the standard line graph with these labels.

Proof. Each case is easily checked. See Figure 4 for (ii). □

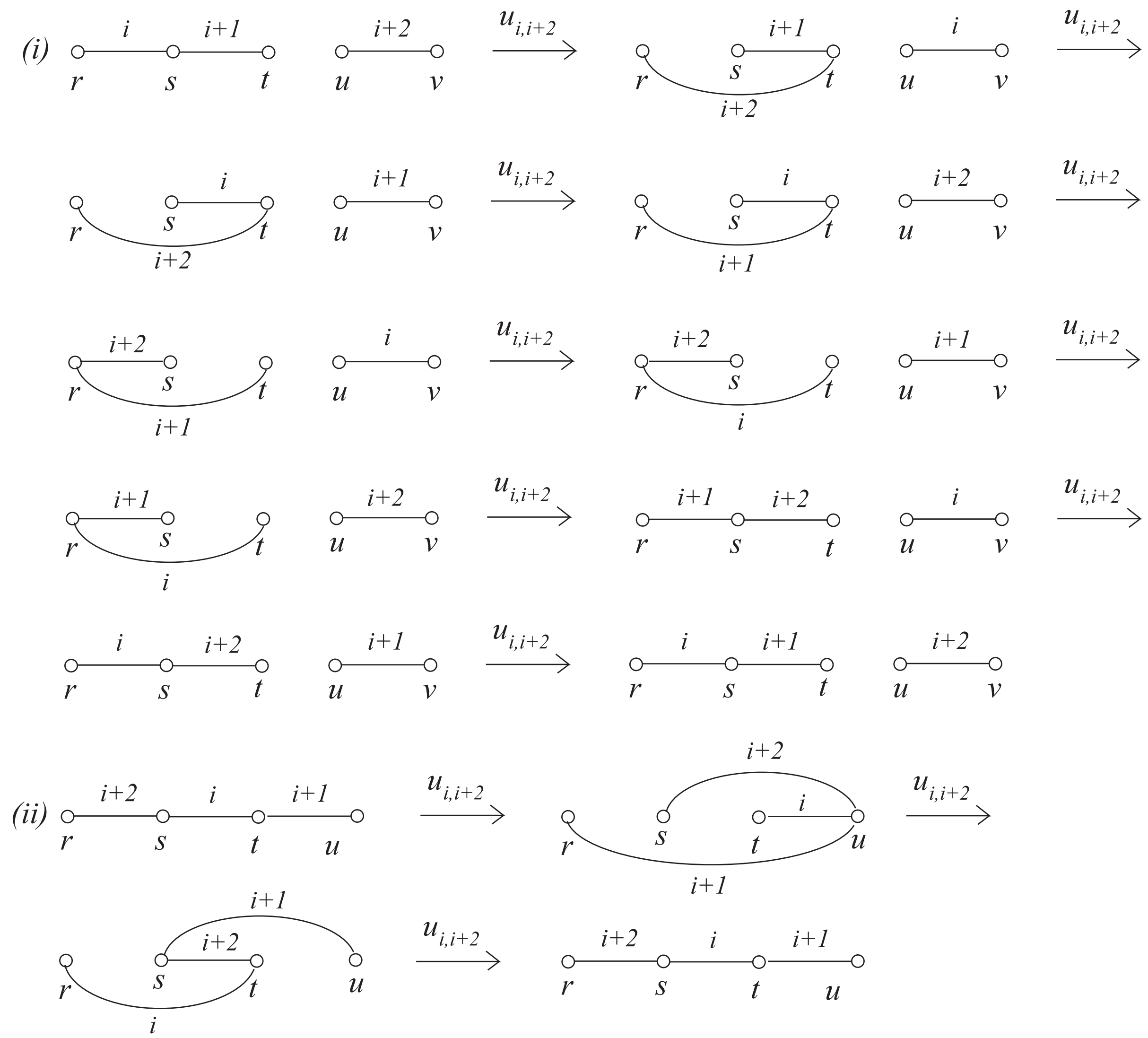

Figure 4.

In order to proceed inductively we first need to show that there is some $\alpha \in H_{n3}$ such that the subgraph $\Gamma(\alpha([S]))_{n-1}$ of $\Gamma(\alpha([S]))$ with edges labeled $1, 2, \ldots, n-1$ is connected. If $\Gamma([S])_{n-1}$ is not connected, then let the two components be $L([S])$ and $R([S])$, where the edge labeled 1 is in $L([S])$. The idea will be to increase the size of $L([S])$. We first show that we can do this if the subgraph $R([S])$ with the nth edge attached is not a line graph with adjacent edges in increasing order.

For $m \leq n$ we will let $e_m = e_m(\Gamma([S]))$ denote the edge of $\Gamma([S])$ that is labeled m.

Now let m be the largest label of an edge in $R([S])$. If $m \neq n-1$, then e_{m+1} is in $L([S])$ and so the two edges e_m, e_{m+1} are disjoint. Thus by Lemma 5.3 (i) if we act by $u_{m,m+1} \in H_{n3}$, then we have increased m. We may thus assume that $m = n-1$.

If e_n, e_{n-1} are not connected (do not form a connected subgraph), then acting by $u_{n-1,n}$ permutes the labels on e_n, e_{n-1} and so increases the size of $L([S])$. So now assume that e_n, e_{n-1} are connected.

Now let m be the largest label of an edge of $R \setminus \{e_{n-1}\}$, so that $m < n-1$. Then, as in the above, we may reduce to the case $m = n-2$. If now e_{n-2} is not connected to $e_n \cup e_{n-1}$, then acting by $u^3_{n-2,n} \in H_{n3}$ will increase the size of $L([S])$ (see Figure 4). If $e_n \cup e_{n-1} \cup e_{n-2}$ has a trivalent vertex, then acting by $u_{n-2,n}$ will increase the size of L (see Figure 4). This deals with all the cases where e_n, e_{n-1}, e_{n-2} is not a line graph. So we now assume that e_n, e_{n-1}, e_{n-2} is a line graph.

We continue this process showing at each stage that we can either increase the size of $L([S])$ or that $e_n, e_{n-1}, \ldots, e_k$ are all in $R([S])$ and form a line graph. We indicate how one does this. So suppose that $e_n, e_{n-1}, \ldots, e_k$ are all in $R([S])$ and that they form a line graph. Assume that there are edges in $R([S])$ other than these and let m be the largest label of such an edge. Note that $m < k$. We show that we may assume that $m = k-1$. For if not, then e_{k-1} is in $L([S])$ and acting by $u_{k-1,k}$ will increase m.

Thus we assume that $m = k-1$. Now if e_{k-1} is disjoint from all of $e_n, e_{n-1}, \ldots, e_k$, then acting by $u_{n-1,n}u_{n-2,n-1}\cdots u_{k-1,k}$ will increase the size of $L([S])$ (this action only changes the labels on the graph, the label of the edge e_n being changed to $k-1$). If e_{k-1} is not disjoint from all of $e_n, e_{n-1}, \ldots, e_k$, but together they do not form a line graph, then (since all these graphs are trees) there is a unique p with $n \geq p > k$ such that e_{k-1} meets e_p and e_{p-1}. Again (using Figure 4) we see that acting by $u_{n-1,n}u_{n-3,n-1}\cdots u_{k,k+2}u_{k-1,k+1}$ will increase the size of $L([S])$ (the edge labeled $n-1$ in the resulting graph is now a part of the resulting L). This completes the proof that either we can increase the size of $L([S])$ or that $R([S])$ is a line graph as indicated. We now deal with this latter case.

So suppose that $R([S])$ is a line graph with edges labeled $n, n-1, \ldots, m, m > 1$. Then e_{m-1} is an edge of $L([S])$. Suppose first that e_{m-1} is disjoint from $e_n \cup e_{n-1} \cup \cdots \cup e_m$. Then using Figure 4 we see that acting by

$$u^3_{n-2,n}u_{n-3,n-1}u_{n-4,n-2}\cdots u_{m-1,m+1}$$

has the effect of increasing the size of $L([S])$. So now we may assume that e_{m-1} is not disjoint from $e_n, e_{n-1}, \ldots, e_m$. Thus e_{m-1} must be adjacent to e_n and we have a line subgraph $e_{m-1}, e_n, e_{n-1}, \ldots, e_m$. Here we first act by $u_{n-1,n-2}u_{n-2,n-3}\cdots u_{m-1,m}$ so as to obtain the line subgraph $e_{n-1}, e_n, e_{n-2}, e_{n-3}, \ldots, e_{m-1}$. One now acts on this by $u^2_{n-2,n}$ and notes (using Figure 4) that we have increased the size of $L([S])$.

Thus we may now assume that the subgraph $e_1, e_2, \ldots, e_{n-1}$ is connected. Hence by induction we assume that the subgraph $e_1, e_2, \ldots, e_{n-1}$ is a star graph with central vertex v say. We consider the following possibilities:

(i) v is a vertex of e_n;
(ii) e_n is adjacent to e_{n-1};
(iii) e_n is adjacent to e_k with $k \geq 3$;
(iv) e_n is adjacent to e_k with $k = 1, 2$.

If we have (i), then we are done. If (ii), then one checks that

$$u_{n-3,n-1}^2 u_{n-2,n} u_{n-2,n-1} u_{n-2,n-1} u_{n-3,n-1} \in H_{n3}$$

transforms this case to the star graph.

If (iii), then we first act by $u_{k+1,k+2} u_{k+2,k+3} \cdots u_{n-2,n-1} u_{n-1,n}$, which does nothing except permute the edge labels so that the the edge that was labeled n is now labeled $k+1$. Now proceed as in (ii), acting by

$$u_{k-2,k}^2 u_{k-1,k+1} u_{k-1,k} u_{k-1,k} u_{k-2,k} \in H_{n3}.$$

This is possible since $k \geq 3$.

If (iv) and e_n is adjacent to e_1, then we act by $u_{24} u_{13}^2 u_{24}^3 u_{23} \in H_{n3}$. If (iv) and e_n is adjacent to e_2, then we act by $u_{24} u_{34} u_{13} u_{23} u_{13} u_{24} \in H_{n3}$.

This completes the proof of Proposition 5.1. □

Thus the image of $\mathcal{B}_n$ is a 2-transitive group of degree $(n+1)^{n-2}$.

A computer calculation using MAGMA [MA] shows that (i) for $n = 3$ we get 4 classes of edge-labeled graphs, the action of $\mathcal{B}_3$ being that of the alternating group A_4; (ii) for $n = 4$ we get 25 classes of edge-labeled graphs, the action of $\mathcal{B}_4$ being that of the symmetric group S_{25}; (iii) for $n = 5$ we get 216 classes of edge-labeled graphs, the action of $\mathcal{B}_5$ being that of the alternating group A_{216}; (iv) for $n = 6$ we get 2401 classes of edge-labeled graphs, the action of $\mathcal{B}_6$ being that of the alternating group A_{2401}. So now assume that $n \geq 7$.

Now 2-transitive groups have been classified; see [Cam pp. 194-197] for a description of the possibilities. Apart from a few cases of low degree (less than or equal to 276), which we need not consider since we are assuming that $n > 6$, (so that $(n+1)^{n-2} > 2401$), the possible degrees of such a 2-transitive group are n for the alternating and symmetric groups, together with

$$q, 2^{4d+2}+1, q^3+1, \frac{q^d-1}{q-1} \text{ (where } d \geq 2), 2^{d-1}(2^d \pm 1) \text{ (where } d \geq 3).$$

Here q is a prime power.

We now show that $n^{n-3} = 1 + p^s$ is not possible if $n > 4$ and p is a prime. For if $n^{n-3} - 1 = p^s$, then $n - 1 = p^r$ and so we have $(p^r+1)^{p^r-2} = 1 + p^s$, where $s \geq 2r$ since $n > 4$. Expanding the left hand side using the binomial theorem we get

$$p^{2r} X + (p^r - 2)p^r + 1 = 1 + p^s.$$

Dividing by p^r we get $p^r X + (p^r - 2) = p^{s-r}$. Now since $s \geq 2r$, we see that p^r divides 2, which gives $n \leq 3$, a contradiction. This eliminates the cases $2^{4d+2}+1, q^3+1$ from the above list.

Let

$$H_{n4} =< \{u_{i,i+1}\}_{1 \leq i < n}, \{u_{i,i+2}^3\}_{1 \leq i < n-1}, \{u_{i,i+3}^4\}_{1 \leq i < n-2} > .$$

Proposition 5.4. *The elements of H_{n4} fix $\Gamma(St_n)$ and Γ_0.*

Proof. One can easily check this statement for each generator of H_{n4}. In fact one need only check the cases $i = 1$. The result follows. $\square$

We conjecture that H_{n4} acts transitively on $PO(A_n) \setminus \{\Gamma_0, \Gamma(ST_n)\}$. This would show that $\rho_n(\mathcal{B}_n)$ is 3-transitive for all $n \geq 3$.

Let $n \geq 4$. We now investigate whether the image of $\mathcal{B}_n$ is the alternating group or the symmetric group in the cases considered in Theorem 1.4. Note that each of the generators $\sigma_i, i < n$, is conjugate to σ_1. Thus we need only check whether the image of σ_{n-1}, say δ_{n-1}, is an even or an odd permutation. Now note that the effect of δ_{n-1} on an edge-labeled graph Γ is to permute the labels $n-1, n$ on the edges labeled $n-1, n$ if those edges are disjoint, and is otherwise the identity. Now it is easy to check that since $n \geq 4$, δ_{n-1} cannot fix any graph Γ where the edges labeled $n-1$ and n are disjoint; this shows that number of such graphs is even. Thus δ_{n-1} is an even permutation if and only if the number of two cycles in δ_{n-1} is a multiple of 2 if and only if the number of graphs Γ with the edges labeled $n-1$ and n disjoint is a multiple of 4. Denote this latter number by ζ_n.

We now calculate the number ζ'_n of graphs Γ which have the property that the edges labeled $n-1$ and n are adjacent. Then we have $(n+1)^{n-2} = \zeta_n + \zeta'_n$, which gives the first statement of

Lemma 5.5. *For $n \geq 4$ we have $\zeta_n = (n+1)^{n-2} - \zeta'_n$ and*

$$\zeta'_n = \sum_{\substack{a,b,c \geq 0 \\ a+b+c=n-2}} \binom{n-2}{a}\binom{n-2-a}{b}(a+1)^{a-1}(b+1)^{b-1}(c+1)^{v-1}.$$

(Here the convention is that $x^y = 1$ if $y < 0$.)

We have

$$\zeta'_n \equiv 0 \ mod\ 4 \ \text{if}\ n \equiv 1, 3 \ mod\ 4;$$
$$\zeta'_n \equiv 3 \ mod\ 4 \ \text{if}\ n \equiv 0 \ mod\ 4;$$
$$\zeta'_n \equiv 1 \ mod\ 4 \ \text{if}\ n \equiv 2 \ mod\ 4.$$

Proof. Let Γ be a graph with n edges labeled $1, 2, \ldots, n$ and let e_i be the edge of Γ labeled i. Assume that e_{n-1} and e_n are adjacent in Γ. Let u, v be the vertices of e_{n-1} and v, w the vertices of e_n. Let A be the component of $\Gamma \setminus \{u\}$ that does not contain e_{n-1}; let B be the component of $\Gamma \setminus \{v\}$ that does not contain e_{n-1}; let C be the component of $\Gamma \setminus \{w\}$ that does not contain e_{n-1}. Let a denote the number of edges of A, b denote the number of edges of B, and c the number of edges of C. Then we have $a+b+c = n-2$ where $a, b, c \geq 0$. The possibilities for Γ come from the possibilities for each of A, B, C. We will think of A, B, C as rooted trees with roots u, v, w and numbers of edges a, b, c edges respectively. Since A is a tree and has a edges, there are $\binom{n-2}{a}$ ways of choosing the labels for the edges of A. Further, for a given such choice of labels for the edges of A there are $(a+1)^{a-1}$ rooted trees [S] and so there are $\binom{n-2}{a}(a+1)^{a-1}$ total possibilities for A.

Given such a choice of labels for A there are then $\binom{n-2-a}{b}$ choices for a set of labels for the edges of B and so $\binom{n-2-a}{b}(b+1)^{b-1}$ total possibilities for B. Having chosen labels for

A, B, the set of labels for C is now determined since $a+b+c=n-2$. Thus there are $(c+1)^{c-1}$ possibilities for C. All of these choices are independent and so the formula for ζ'_n given in Lemma 5.5 follows.

Assume that $n>5$ is odd. We consider each choice of $a, b, c \geq 0$ where $a+b+c=n-2$. Then at least one of a, b, c is odd. If a is odd and $a \geq 3$, then $(a+1)^{a-1} \equiv 0 \bmod 4$. Similarly for b, c. Thus we may assume that either a is even or $a<3$ and similarly for b, c. One now checks that all possible choices of a, b, c give a summand of ζ'_n which is a multiple of 4. Thus $\zeta'_n \equiv 0 \bmod 4$. The rest of the cases are similar. One uses the fact that if $n \equiv 2 \bmod 4$ and a is odd, then $\binom{n-2}{a} \equiv 0 \bmod 4$ and $\binom{n}{a} \equiv 0 \bmod 2$. $\square$

Corollary 5.6. *Let $n>3$. Then*

$$\begin{aligned}
&\zeta_n \equiv 0 \textit{ if } n \equiv 1,2,3 \textit{ mod } 4; \\
&\zeta_n \equiv 2 \textit{ if } n \equiv 0 \textit{ mod } 4. \quad \square
\end{aligned}$$

§6 The $PO(B_n)$ case

In this section we prove Theorem 1.5.

Recall that each element of $PO(B_n)$ is a conjugacy class of some sequence $S=(s_1, \ldots, s_n)$ of n reflections generating $W(B_n)$. These reflections are the γ_i and the $\beta_{rs}^{\pm}$ and in S we have exactly one of the γ_i. Since no γ_i is conjugate to a $\beta_{rs}^{\pm}$, the position of this γ_i in S is independent of which $W(B_n)$-conjugate of S one chooses; denote this position by $\tau(S)$. Let R_u be the set of all $[S] \in PO(B_n)$ such that $\tau(S)=u$. Then $R_1, \ldots, R_n$ is a set of blocks for the $\mathcal{B}_n$ action on $PO(B_n)$. Further it is easily seen that $\sigma_i(R_j)=R_{(i,i+1)j}$ and so the action of $\mathcal{B}_n$ on these blocks is that of the full symmetric group S_n.

We now show that the degree of this representation is n^{n-1}. As in the proof of the B_n case in §3 we note that any set $S=(s_1, \ldots, s_n)$ of generating reflections in the B_n case can be conjugated to the form $(\delta_1, \ldots, \delta_n)$, where $n-1$ of the δ_i have the form β_{rs}^{+} and the remaining one is γ_j. Further, it is easy to see that this representative of the conjugacy class of S is unique. Thus removing the γ_j from the sequence S results in a set S' of $n-1$ reflections which generate S_n. We have seen in Theorem 1.4 that (up to conjugacy) there are n^{n-3} of these. Thus the number of elements of $PO(B_n)$ will be equal to n^{n-3} multiplied by the number of ways of inserting the γ_j into S'. Since there are n places to put the γ_j and there are n choices of j, we thus see that the number of elements of $PO(B_n)$ is n^{n-1}, as required. This completes the proof of Theorem 1.5 $\square$

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