

Prove that the matrix :

14

$$A = \begin{bmatrix} 5 & 2 & 3 \\ 6 & 4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$$

has a determinant of zero.

Answer :

If we switch the first and third row we call the resultant^{ing} matrix B .

according to theorem, $\det(A) = -\det(B)$

however, since all the entries of $B =$ entries of A , $B = A$.

thus, $\det(A) = \det(B)$.

so, $\det(A) = -\det(B) = -\det(A) = \det(B)$.

since $\det(A) = -\det(A)$, $\det(A) = 0$.

} little more
efficient

best
dialogue
in
exam,
but could
be
better

6

Math 343 Exam Questions

Question #1:

Points: 40

Use Cramer's rule to solve

$$x_1 + 2x_2 + x_3 = 5$$

$$2x_1 + 2x_2 + x_3 = 6$$

$$x_1 + 2x_2 + 3x_3 = 9$$

Solution:

Because A is a square nonsingular matrix, it is appropriate to apply Cramer's rule.

Let A_i be the matrix obtained by replacing the i th column of A by b . Through Cramer's rule, we know that x_i can be obtained by $\det(A_i) / \det(A)$.

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -4$$

$$\det(A_1) = \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = -4$$

good

$$\det(A_2) = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = -4$$

$$\det(A_3) = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = -8$$

Therefore,

$$x_1 = (-4) / (-4) = 1$$

$$x_2 = (-4) / (-4) = 1$$

$$x_3 = (-8) / (-4) = 2$$

Question #2:

Total Points: 40

Theorem: Suppose A is an invertible matrix, then $Ax = b$ has exactly one solution, namely $x = A^{-1}b$.

a) 25 points: Prove the Theorem

b) 15 points: Apply it to $x + y - 2z = 12$

$$X + y - \frac{1}{2}z = 7$$

$$X - 4y + \frac{1}{2}z = -3$$

Solutions:

a) If $Ax = b$ has a solution and A inverse exists, then the solution is $A^{-1}b = x$. Then $Ax = b$ implies that $A^{-1}(Ax) = A^{-1}b$ implies that $(A^{-1}A)x = A^{-1}b$ because matrix multiplication is associative. $Ix = A^{-1}b$ because we know $(A^{-1}A)$ is equal to the identity matrix; then $x = A^{-1}b$. Moreover, if $x = A^{-1}b$, then $Ax = A(A^{-1}b)$ which implies that $Ax = (AA^{-1})b = Ib = b$.

b) The coefficient matrix of the system of equations is:

$$\left| \begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & 0.5 & 0 & 1 & 0 \\ 1 & -4 & 0.5 & 0 & 0 & 1 \end{array} \right| \begin{array}{l} 2R2 \rightarrow R2 \\ 2R3 \rightarrow R3 \end{array} \left| \begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 \\ 2 & -8 & 1 & 0 & 0 & 2 \end{array} \right| \begin{array}{l} R2-2R1 \rightarrow R2 \\ R3-2R1 \rightarrow R3 \end{array}$$

$$\left| \begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 5 & -2 & 2 & 0 \\ 0 & 10 & 5 & -2 & 0 & 2 \end{array} \right| \begin{array}{l} R3 \leftrightarrow R2 \end{array} \left| \begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 10 & 5 & -2 & 0 & 2 \\ 0 & 0 & 5 & -2 & 2 & 0 \end{array} \right| \begin{array}{l} (-1/10)R2 \rightarrow R2 \\ (1/5)R3 \rightarrow R3 \end{array}$$

$$\left| \begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0.5 & 0.2 & 0 & -0 \\ 0 & 0 & 1 & -0 & 0.4 & 0 \end{array} \right| \begin{array}{l} R2 + .5R3 \rightarrow R2 \\ R1 + 2 \\ R3 \rightarrow R1 \end{array} \left| \begin{array}{ccc|ccc} 1 & 1 & 0 & 0.2 & 0.8 & 0 \\ 0 & 1 & 0 & 0 & 0.2 & -0 \\ 0 & 0 & 1 & -0 & 0.4 & 0 \end{array} \right| \begin{array}{l} R1 - R2 \rightarrow R1 \end{array}$$

$$\left| \begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.6 & 0.2 \\ 0 & 1 & 0 & 0 & 0.2 & -0 \\ 0 & 0 & 1 & -0 & 0.4 & 0 \end{array} \right|$$

The right hand side of the matrix is the inverse of the coefficient matrix, so we have

$$A^{-1} = \left| \begin{array}{ccc} 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & -0 \\ -0 & 0.4 & 0 \end{array} \right|$$

$$\text{And } b = \left| \begin{array}{c} 12 \\ 7 \\ -3 \end{array} \right|$$

$$\text{Making } A^{-1}b = \left| \begin{array}{ccc} 0.2 & 0.6 & 0.2 \\ 0 & 0.2 & -0 \\ -0 & 0.4 & 0 \end{array} \right| * \left| \begin{array}{c} 12 \\ 7 \\ -3 \end{array} \right| = \left| \begin{array}{c} 6 \\ 2 \\ -2 \end{array} \right|$$

So the solution is

$$\left| \begin{array}{c} x \\ y \\ z \end{array} \right| = \left| \begin{array}{c} 6 \\ 2 \\ -2 \end{array} \right|$$

Or $x = 6, y = 2, z = -2$

Question #3:

Points: 30

Show that if matrix $B =$ matrix A , except that one of B 's rows has been multiplied by k , then $\det(B) = k \cdot \det(A)$.

to produce the ~~matrix~~ row of B,

Solution:

$$\begin{aligned}\det(B) &= \sum_{P: [n] \rightarrow [n]} (-1)^{d(P)} b_{1P(1)} * \dots * b_{jP(j)} * \dots * b_{nP(n)} \\ &= \sum_{P: [n] \rightarrow [n]} (-1)^{d(P)} a_{1P(1)} * \dots * k a_{jP(j)} * \dots * a_{nP(n)} \\ &= k \sum_{P: [n] \rightarrow [n]} (-1)^{d(P)} a_{1P(1)} * \dots * a_{jP(j)} * \dots * a_{nP(n)} \\ &= k * \det(A)\end{aligned}$$

great

Question #4:

Points: 40

Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$ using elementary row operations

Solution:

$$\begin{aligned}\det(A) &= \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} \xrightarrow{R_2 - 4R_1 \rightarrow R_2} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 10 \end{vmatrix} \xrightarrow{R_3 - 7R_1 \rightarrow R_3} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{vmatrix} \xrightarrow{\left(\frac{-1}{3}\right)R_2 \rightarrow R_2} \\ &= (-3) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -11 \end{vmatrix} \xrightarrow{R_3 + 6R_2 \rightarrow R_3} (-3) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = (-3)(1)(1)(1) = -3\end{aligned}$$

good

2

1) Find the determinant of

$$\begin{bmatrix} 2 & -5 & 7 & 9 & -4 \\ 2 & 3 & 9 & 6 & 10 \\ 4 & 10 & 13 & 11 & 13 \\ 2 & 3 & 6 & 1 & 4 \\ -2 & 1 & -8 & -4 & -2 \end{bmatrix}$$

SOLUTION:

$$\begin{aligned} & \begin{vmatrix} 2 & -5 & 7 & 9 & -4 \\ 2 & 3 & 9 & 6 & 10 \\ 4 & 10 & 13 & 11 & 13 \\ 2 & 3 & 6 & 1 & 4 \\ -2 & 1 & -8 & -4 & -2 \end{vmatrix} R2 \leftrightarrow R3, \\ \Rightarrow & \begin{vmatrix} 2 & -5 & 7 & 9 & -4 \\ 4 & 10 & 13 & 11 & 13 \\ 2 & 3 & 9 & 6 & 10 \\ 2 & 3 & 6 & 1 & 4 \\ -2 & 1 & -8 & -4 & -2 \end{vmatrix} R5 + R3 \rightarrow R5, R2 - 2R4 \rightarrow R2, R1 - R3 \rightarrow R1 \\ \Rightarrow & \begin{vmatrix} 0 & -8 & -2 & 3 & -14 \\ 0 & 4 & 1 & 9 & 5 \\ -2 & 3 & 9 & 6 & 10 \\ 2 & 3 & 6 & 1 & 4 \\ 0 & 4 & 1 & 2 & 8 \end{vmatrix} R1 + 2R5 \rightarrow R1, R4 \leftrightarrow R1 \\ \Rightarrow & \begin{vmatrix} 2 & 3 & 6 & 1 & 4 \\ 0 & 4 & 1 & 9 & 5 \\ 2 & 3 & 9 & 6 & 10 \\ 0 & 0 & 0 & 7 & 2 \\ 0 & 4 & 1 & 2 & 8 \end{vmatrix} R2 - R4 \rightarrow R2, R5 - R2 \rightarrow R5, R3 - R1 \rightarrow R3 \end{aligned}$$

$$\Rightarrow \begin{vmatrix} 2 & 3 & 6 & 1 & 4 \\ 0 & 4 & 1 & 2 & 3 \\ 0 & 0 & 3 & 5 & 6 \\ 0 & 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix}, \text{ and using a theorem, such that the determinant of a triangular matrix is equal to the product of its diagonal entries:}$$

$$\Rightarrow \begin{vmatrix} 2 & 3 & 6 & 1 & 4 \\ 0 & 4 & 1 & 2 & 3 \\ 0 & 0 & 3 & 5 & 6 \\ 0 & 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix} = 2 \cdot 4 \cdot 3 \cdot 7 \cdot 5 = 840, \text{ Therefore the determinant of the matrix is 840.}$$

(35 Points)

2) What conditions must b_1 , b_2 , and b_3 satisfy in order for the system equations

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= b_1 \\ x_1 + x_3 &= b_2 \\ 2x_1 + 2x_2 + 4x_3 &= b_3 \end{aligned}$$

to be consistent? Use the resulting formulas to find the solution if $b_1 = 1$, $b_2 = 3$, $b_3 = -1$

SOLUTION:

$$\begin{bmatrix} 1 & 2 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 2 & 4 & b_3 \end{bmatrix} \begin{matrix} \\ R2 - R1 \rightarrow R2, R3 - 2R1 \rightarrow R3 \\ \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & b_1 \\ 0 & -2 & -1 & b_2 - b_1 \\ 0 & -2 & 0 & b_3 - 2b_1 \end{bmatrix} \begin{matrix} \\ R2 \leftrightarrow R3, R2 * -1/2 \rightarrow R2 \\ \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & b_1 \\ 0 & 1 & 0 & (-1/2)b_3 + b_1 \\ 0 & -2 & -1 & b_2 - b_1 \end{bmatrix} \begin{matrix} \\ R3 + 2R2 \rightarrow R3, -R3 \rightarrow R3 \\ \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & b_1 \\ 0 & 1 & 0 & (-1/2)b_3 + b_1 \\ 0 & 0 & 1 & b_3 - b_2 - b_1 \end{bmatrix} \begin{matrix} \\ R1 - 2R2 \rightarrow R1, R1 - 2R3 \rightarrow R1 \\ \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & b_1 + 2b_2 - b_3 \\ 0 & 1 & 0 & (-1/2)b_3 + b_1 \\ 0 & 0 & 1 & b_3 - b_2 - b_1 \end{bmatrix}$$

- you never answer this question

do these give consistency?

$$\Rightarrow \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And understanding that the determinant of a diagonal matrix is equal to the product of the entries on the diagonal, $\det(\mathbf{AxB}) = 1$

Therefore, as $\det(\mathbf{AxB}) = 1$, $\det(\mathbf{A}) \times \det(\mathbf{B}) = 1$

(30 Points)

4) Assuming matrix $\mathbf{A}$ is invertible, prove that the reduced row-echelon form of $\mathbf{A}$ is $\mathbf{I}_n$.

SOLUTION:

Let $\mathbf{Ax} = \mathbf{0}$, and assuming $\mathbf{A}$ is invertible,

$$\mathbf{Ax} = \mathbf{0} \Rightarrow \mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}\mathbf{0}$$

And because matrix multiplication is associative

$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} \Rightarrow \mathbf{Ix} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$$

Which means that $\mathbf{Ax} = \mathbf{0}$, only has the trivial solution, $\mathbf{x} = \mathbf{0}$, which is shown as

$$\begin{array}{rcl} x_1 & & = 0 \\ x_2 & & = 0 \\ & \dots & \\ & \dots & \\ x_n & & = 0 \end{array}$$

- good

Which makes $\mathbf{Ax} = \mathbf{0}$ comparable to

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Which makes it evident that $\mathbf{A}$ can be reduced to the identity matrix, $\mathbf{I}_n$. As the only non-zero numbers in the matrix are on the diagonal, this is the reduced row-echelon form of $\mathbf{A}$, which is $\mathbf{I}_n$.

(40 Points)

10

#1

Question:

a) For what values of x or y is A singular and why?

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ x & 2 & 6 & 2 \\ 0 & 0 & y & 0 \\ 3 & 2 & -2 & 1 \end{bmatrix}$$

b) Set $x=1$ and $y=3$. Find the inverse of the matrix A using row operations.

Solution:

a) Either $x=4$ or $y=0$ makes A singular. In the case $y=0$ the entries in $R3$ are all zero. In the case of $x=4$ $R2=2*R1$. This means you could reduce this to having a row of zeros. Anytime a matrix has a row of zeros, that matrix is singular. It does not have an inverse.

b)

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 6 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\ 3 & 2 & -2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R1 \longleftrightarrow R2$$

$$1/3 * R3 \longrightarrow R3$$

$$-R_2 + R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & 2 & 6 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & -4 & -20 & -7 & 0 & -3 & 0 & 1 \end{bmatrix}$$

- do you know
how to
postpone
fractions?

$$-6R_3 + R_1 \rightarrow R_1$$

$$-2R_1 + R_2 \rightarrow R_2$$

$$R_4 * -1 \rightarrow R_4$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 0 & 1 & -2 & 0 \\ 0 & -3 & -9 & -3 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 4 & 20 & 7 & 0 & 3 & 0 & -1 \end{bmatrix}$$

$$R_4 + R_2 \rightarrow R_2$$

$$\frac{4}{3}R_2 + R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 0 & 1 & -2 & 0 \\ 0 & 1 & 11 & 4 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 8 & 3 & \frac{4}{3} & \frac{1}{3} & 0 & -1 \end{bmatrix}$$

$$-11R_3 + R_2 \rightarrow R_2$$

$$-8R_3 + R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 4 & 1 & 1 & \frac{-11}{3} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 3 & \frac{4}{3} & \frac{1}{3} & \frac{-8}{3} & -1 \end{bmatrix}$$

$$-2R_2 + R_1 \rightarrow R_1$$

$$\frac{1}{3}R_4 \rightarrow R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & -6 & -2 & -1 & \frac{16}{3} & 2 \\ 0 & 1 & 0 & 4 & 1 & 1 & \frac{-11}{3} & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{4}{9} & \frac{1}{9} & \frac{-8}{9} & \frac{-1}{3} \end{bmatrix}$$

$$6R_4 + R_1 \rightarrow R_1$$

$$-4R_4 + R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{2}{3} & \frac{-1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{-7}{9} & \frac{5}{9} & \frac{-1}{9} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{4}{9} & \frac{1}{9} & \frac{-8}{9} & \frac{-1}{3} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} & 0 & 0 \\ \frac{-7}{9} & \frac{5}{9} & \frac{-1}{9} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 \\ \frac{4}{9} & \frac{1}{9} & \frac{-8}{9} & \frac{-1}{3} \end{bmatrix}$$

#2

Given:

$$\text{Matrix A} = \begin{bmatrix} 0 & 1 & 0 \\ 4 & -8 & 12 \\ 3 & 2 & 4 \end{bmatrix}$$

Problem: Find the value of the determinant of the given matrix by the following methods:

- a) Cofactor Expansion
- b) Row Reduction
- c) Repeating the first two columns

Show your work!

Solution:

a) Along the first row:

$$\det(A) = 0 \begin{vmatrix} -8 & 12 \\ 2 & 4 \end{vmatrix} - 1 \begin{vmatrix} 4 & 12 \\ 3 & 4 \end{vmatrix} + 0 \begin{vmatrix} 4 & -8 \\ 3 & 2 \end{vmatrix} = -1(16-36) = 20$$

b) $\det(A) = \begin{bmatrix} 0 & 1 & 0 \\ 4 & -8 & 12 \\ 3 & 2 & 4 \end{bmatrix}$ $R2 \leftrightarrow R1$ and take common
factor 4 from R2 through the determinant sign.

$$= -4 \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 3 & 2 & 4 \end{bmatrix} \xrightarrow{R3-3R1 \rightarrow R3} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 8 & -5 \end{bmatrix} \xrightarrow{R3-8R2 \rightarrow R3}$$

$$= -4 \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix} = -4(1)(1)(-5) = 20$$

$$\begin{array}{ccccc} & [0 & 1 & 0] & 0 & 1 \\ \text{c)} & [4 & -8 & 12] & 4 & -8 \\ & [3 & 2 & 4] & 3 & 2 \end{array}$$

The determinant is then found by adding the products on the diagonals starting in the top left corner with zero and ending with 4, and the next two diagonals to the right of that first one, and then subtracting the products on the diagonals going in the opposite direction starting at the top right corner with 1 and ending with 4 and the next two diagonals to the left of that one.

$$\begin{aligned} \text{Thus } \det(A) &= 0(-8)(4) + 1(12)(3) + 0(4)(2) - 1(4)(4) - \\ & 0(12)(2) - 0(-8)(3) = 36 - 16 = 20 \end{aligned}$$

#3

State whether the following statements are true or false and give an explanation of why. (.5 pts. Each)

a.) Matrices A and B are commutable.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

b.) If $A + B$ is symmetric, then so are A and B .

c.) In theory, Cramer's rule can be used to solve any system of linear equations, although the amount of computation may be enormous.

d.) If AB is invertible, then A and B must also be invertible.

Solution:

a.) True, if A and B are matrices such that $AB=BA$ (thus they are symmetric), then we say that A and B commute.

? logic?

$$AB = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$$

b.) False, $A + B$ can be symmetric without A and B being symmetric. However, if A and B are symmetric then following theorem 1.7.2, $A + B$ must be symmetric.

must be in n unknowns

c.) True, for n systems of linear equations it is possible to use Cramer's rule but it would be better to use Gaussian elimination to solve the equations.

Also, must be invertible matrix.

d.) True if A and B are square matrices then the statement is true. If they are not square matrices then the statement is false.

don't say this before 0.6.

perfect

Manipulate the following matrix to reduced row-echelon form:

$$\begin{bmatrix} 6 & 3 & 12 & 1 & 7 \\ 0 & -4 & 12 & 0 & 0 \\ 0 & -2 & 4 & 0 & -6 \\ -6 & 8 & 4 & 0 & -3 \end{bmatrix}$$

Solution:

$$R_1 + R_4 \rightarrow R_1 \quad \begin{bmatrix} 0 & -5 & 16 & 1 & 7 \\ 0 & -4 & 12 & 0 & 0 \\ 0 & -2 & 4 & 0 & -6 \\ -6 & -8 & 4 & 0 & -3 \end{bmatrix}$$

$$\begin{array}{l} R_2 - 3R_3 \rightarrow R_2 \\ R_1 \leftrightarrow R_4 \end{array} \quad \begin{bmatrix} -6 & -8 & 4 & 0 & -3 \\ 0 & 2 & 0 & 0 & 18 \\ 0 & -2 & 4 & 0 & -6 \\ 0 & -5 & 16 & 1 & 7 \end{bmatrix}$$

$$R_3 + 6R_1 \rightarrow R_3 \quad \begin{bmatrix} 1 & -1 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 & 18 \\ 0 & -2 & 4 & 0 & -6 \\ 0 & -5 & 16 & 1 & 7 \end{bmatrix}$$

$$\begin{array}{l} R_1 + \frac{1}{2}R_2 \rightarrow R_1 \\ R_4 - 4R_3 \rightarrow R_4 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 6 \\ 0 & 2 & 0 & 0 & 18 \\ 0 & -2 & 4 & 0 & -6 \\ 0 & 3 & 0 & 1 & 31 \end{bmatrix}$$

$$\frac{1}{2}R_2 \rightarrow R_2 \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 9 \\ 0 & -2 & 4 & 0 & -6 \\ 0 & 3 & 0 & 1 & 31 \end{bmatrix}$$

$$\begin{array}{l} R_3 + 2R_2 \rightarrow R_3 \\ R_4 - 3R_2 \rightarrow R_4 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 9 \\ 0 & 0 & 4 & 0 & 12 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\frac{1}{4}R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

R₁?

I don't see this

maybe

1. Solve the following system of equations by using Gauss-Jordan Elimination.

11

$$\begin{aligned}a+2b-c-2d &= -1 \\ 2a+b+c-d &= 4 \\ a+b+2c+d &= 5 \\ a+3b+2c-3d &= -3\end{aligned}$$

Solution:
(30 Points)

Step 1:

$$\begin{bmatrix} 1 & 2 & -1 & -2 & -1 \\ 2 & 1 & 1 & -1 & 4 \\ 1 & 1 & 2 & 1 & 5 \\ 1 & 3 & 2 & -3 & -3 \end{bmatrix} \quad \text{Subtract row 1 from row 3}$$

Step 2:

$$\begin{bmatrix} 1 & 2 & -1 & -2 & -1 \\ 2 & 1 & 1 & -1 & 4 \\ 1 & 1 & 2 & 1 & 5 \\ 1 & 1 & 3 & -1 & -2 \end{bmatrix} \quad \text{Subtract row 3 from rows 1 and 2}$$

Step 3:

$$\begin{bmatrix} 0 & 1 & -3 & -3 & -6 \\ 1 & 0 & -1 & -2 & -1 \\ 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & -1 & -2 \end{bmatrix} \quad \text{Subtract row 2 from row 3}$$

Step 4:

$$\begin{bmatrix} 0 & 1 & -3 & -3 & -6 \\ 1 & 0 & -1 & -2 & -1 \\ 0 & 1 & 3 & 3 & 6 \\ 0 & 1 & 3 & -1 & -2 \end{bmatrix} \quad \text{Subtract row 4 from rows 3 and 1}$$

Step 5:

$$\begin{bmatrix} 0 & 0 & -6 & -2 & -4 \\ 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 1 & 3 & -1 & -2 \end{bmatrix} \quad \text{Divide row 1 by 2. Multiply row 1 by -1. Add row 1 to row 4.}$$

Step 6:

$$\begin{bmatrix} 0 & 0 & 3 & 1 & 2 \\ 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 1 & 0 & -2 & -4 \end{bmatrix} \quad \text{Divide row 3 by 4}$$

Step 7:

$$\begin{bmatrix} 0 & 0 & 3 & 1 & 2 \\ 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 & -4 \end{bmatrix} \quad \text{Subtract row 3 from row 1. Multiply row 4 by -1. Divide row 1 by 3.}$$

Step 8:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 \end{bmatrix} \quad \text{Add row 1 to row 2. Add 2 times row 3 to row 2.}$$

Step 9:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 \end{bmatrix} \quad \text{Subtract 2 times row 3 from row 4}$$

Step 10:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ Swap rows 1 and 2. Swap rows 3 and 4. Finally swap rows 2 and 3.}$$

Reduced Matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Thus we see that $a = 3$, $b = 0$, $c = 0$, and $d = 2$.

2. Find the inverse of the following matrix by multiplying the matrix by the identity matrix simultaneously? — doesn't mean what you think, it means

$$(A) = \begin{bmatrix} -1 & -1 & -2 \\ 4 & -2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad (\text{Princess Bride})$$

Solution:
(40 Points)

$$(A) = \begin{bmatrix} -1 & -1 & -2 \\ 4 & -2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step1:

$$\begin{bmatrix} -1 & -1 & -2 \\ 0 & -6 & -7 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 4R1 + R2 \Rightarrow R2$$

Step2:

$$\begin{bmatrix} -1 & -1 & -2 \\ 0 & -6 & -7 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad R1 + R3 \Rightarrow R3$$

Step3:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -6 & -7 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad -R1 \Rightarrow R1$$

Step4:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -6 & -7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 4 & 1 & 0 \\ 10 & 1 & 6 \end{bmatrix} \quad 6R3 + R2 \Rightarrow R3$$

Step5:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 10 & 1 & 6 \end{bmatrix} -1R2 \Rightarrow R2$$

Step6:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 66 & 6 & 42 \\ 10 & 1 & 6 \end{bmatrix} 7R3 + R2 \Rightarrow R2$$

Step7:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 19 & 2 & 12 \\ 66 & 6 & 42 \\ 10 & 1 & 6 \end{bmatrix} 2R3 + R1 \Rightarrow R1$$

Step8:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 19 & 2 & 12 \\ 66 & 6 & 42 \\ -10 & -1 & -6 \end{bmatrix} -R3 \Rightarrow R3$$

Step9:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 19 & 2 & 12 \\ 11 & 1 & 7 \\ -10 & -1 & -6 \end{bmatrix} \frac{1}{6}R2 \Rightarrow R2$$

Step10:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 1 & 5 \\ 11 & 1 & 7 \\ -10 & -1 & -6 \end{bmatrix} -R2 + R1 \Rightarrow R1$$

End Solution:

$$A^{(-1)} = \begin{bmatrix} 8 & 1 & 5 \\ 11 & 1 & 7 \\ -10 & -1 & -6 \end{bmatrix}$$

2. (40 Points)

Find the inverse of the following matrix by finding the adjoint.

$$\begin{bmatrix} 3 & 12 & 9 \\ 17 & 4 & 12 \\ 18 & 8 & 11 \end{bmatrix}$$

$$A^{-1} = 1/\det(A) \operatorname{adj}(A)$$

I found the determinate by using the cofactors.

$$\begin{vmatrix} 3 & 12 & 9 \\ 17 & 4 & 12 \\ 18 & 8 & 11 \end{vmatrix} = 3 \begin{vmatrix} 4 & 12 \\ 8 & 11 \end{vmatrix} - 12 \begin{vmatrix} 17 & 12 \\ 18 & 11 \end{vmatrix} + 9 \begin{vmatrix} 17 & 4 \\ 18 & 8 \end{vmatrix}$$

$$= (-156) - (-348) + (576) = 768$$

I then found the matrix of cofactors.

Matrix of cofactors:

$$\begin{bmatrix} -52 & -29 & 64 \\ 60 & -129 & -192 \\ 108 & -117 & -192 \end{bmatrix} \times \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} = \begin{bmatrix} -52 & 29 & 64 \\ -60 & -129 & 192 \\ 108 & 117 & -192 \end{bmatrix}$$

Then Transposing you get the adjoint

The Inverse then is:

$$\begin{bmatrix} -52 & -60 & 108 \\ 29 & -129 & 117 \\ 64 & 192 & -192 \end{bmatrix} \rightarrow \frac{1}{768.0} \begin{bmatrix} -52 & -60 & 108 \\ 29 & -129 & 117 \\ 64 & 192 & -192 \end{bmatrix}$$

4. Given the matrix A as follows

$$A = \begin{bmatrix} 5 & 2 & 0 & X \\ 0 & 3 & -2 & X^2 \\ 0 & X & 0 & 1 \\ 5X & 3 & 0 & X^2 \end{bmatrix}$$

- Find X such that the determinant of matrix A is zero.
- Find the determinant of A when $X = 0$
- Find the determinant of A when $X = 2$

Solution:

(40 Points)

calculating

Solving for the determinant of A by cofactor expansion along the 3rd column gives

$$\text{Det}(A) = -2 \begin{vmatrix} 5 & 2 & X \\ 0 & X & 1 \\ 5X & 3 & X^2 \end{vmatrix}$$

Continuing with cofactor expansion along the 2nd row gives

$$\text{Det}(A) = -2 \left(X \begin{vmatrix} 5 & X \\ 5X & X^2 \end{vmatrix} + 1 \begin{vmatrix} 5 & 2 \\ 5X & 3 \end{vmatrix} \right)$$

Evaluating these determinants gives us the equation

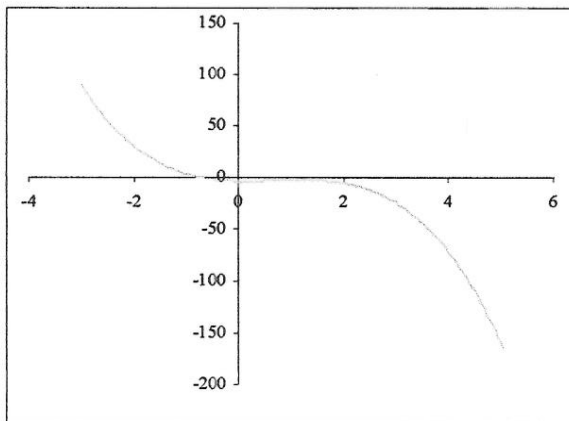
$$\begin{aligned} \text{Det}(A) &= -2X(5X^2 - 5X^2) - 2(15 - 10X) \\ &= -30 + 20X \end{aligned}$$

Part a) Set $\text{Det}(A) = 0$, therefore $0 = -30 + 20X$ thus $X = 3/2$

Part b) Set $X = 0$, therefore $\text{Det}(A) = -30 + 20(0)$ thus $\text{Det}(A) = -30$

Part c) Set $X = 2$, therefore $\text{Det}(A) = -30 + 20(2)$ thus $\text{Det}(A) = 10$

(5)



The curve at the left contains the following points:

$(-2, 30)$, $(-1, 3)$, $(0, -4)$, and $(3, -25)$

What are the coefficients a , b , c , and d such that the curve at the left is given by the following equation?

$$y = ax^3 + bx^2 + cx + d$$

no antecedit and/or non sequitur

Solution:

Use the given points to write each equation in terms of its coefficients:

$$\begin{array}{lcl} (-2, 30) & a(-2)^3 + b(-2)^2 + c(-2) + d = 30 & \\ & -8a + 4b - 2c + d = 30 & \end{array}$$

$$\begin{array}{lcl} (-1, 3) & a(-1)^3 + b(-1)^2 + c(-1) + d = 3 & \\ & -a + b - c + d = 3 & \end{array}$$

$$\begin{array}{lcl} (0, -4) & a(0)^3 + b(0)^2 + c(0) + d = -4 & \\ & 0 + 0 + 0 + d = -4 & \end{array}$$

$$\begin{array}{lcl} (3, -25) & a(3)^3 + b(3)^2 + c(3) + d = -25 & \\ & 27a + 9b + 3c + d = -25 & \end{array}$$

Now that we have a system of four equations and four variables, put the equations in matrix form and then use Gauss-Jordan elimination to put the matrix in reduced row echelon form:

$$\left[\begin{array}{cccc|c} -8 & 4 & -2 & 1 & 30 \\ -1 & 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 1 & -4 \\ 27 & 9 & 3 & 1 & -25 \end{array} \right]$$

$$R_1 \leftrightarrow R_2, R_3 \leftrightarrow R_4$$

$$\begin{array}{ccccc}
 -1 & 1 & -1 & 1 & 3 \\
 -8 & 4 & -2 & 1 & 30 \\
 27 & 9 & 3 & 1 & -25 \\
 0 & 0 & 0 & 1 & -4
 \end{array}
 \quad (-1)R_1$$

$$\begin{array}{ccccc}
 1 & -1 & 1 & -1 & -3 \\
 -8 & 4 & -2 & 1 & 30 \\
 27 & 9 & 3 & 1 & -25 \\
 0 & 0 & 0 & 1 & -4
 \end{array}
 \quad \begin{array}{l} R_2 + 8R_1 \rightarrow R_2 \\ R_3 - 27R_1 \rightarrow R_3 \end{array}$$

$$\begin{array}{ccccc}
 1 & -1 & 1 & -1 & -3 \\
 0 & -4 & 6 & -7 & 6 \\
 0 & 36 & -24 & 28 & 56 \\
 0 & 0 & 0 & 1 & -4
 \end{array}
 \quad R_3 + 9R_4 \rightarrow R_3$$

$$\begin{array}{ccccc}
 1 & -1 & 1 & -1 & -3 \\
 0 & -4 & 6 & -7 & 6 \\
 0 & 0 & 30 & -35 & 110 \\
 0 & 0 & 0 & 1 & -4
 \end{array}
 \quad \begin{array}{l} R_2 / -4 \\ R_3 / 30 \end{array}$$

$$\begin{array}{ccccc}
 1 & -1 & 1 & -1 & -3 \\
 0 & 1 & -3/2 & 7/4 & -3/2 \\
 0 & 0 & 1 & -7/6 & 11/3 \\
 0 & 0 & 0 & 1 & -4
 \end{array}
 \quad \begin{array}{l} R_3 + (7/6)R_4 \rightarrow R_3 \\ R_2 - (7/4)R_4 \rightarrow R_2 \\ R_1 + R_4 \rightarrow R_1 \end{array}$$

$$\begin{array}{ccccc}
 1 & -1 & 1 & 0 & -7 \\
 0 & 1 & -3/2 & 0 & 11/2 \\
 0 & 0 & 1 & 0 & -1 \\
 0 & 0 & 0 & 1 & -4
 \end{array}
 \quad \begin{array}{l} R_2 + (3/2)R_3 \rightarrow R_2 \\ R_1 - R_3 \rightarrow R_1 \end{array}$$

$$\begin{array}{ccccc}
 1 & -1 & 0 & 0 & -6 \\
 0 & 1 & 0 & 0 & 4 \\
 0 & 0 & 1 & 0 & -1 \\
 0 & 0 & 0 & 1 & -4
 \end{array}$$

$$R_1 + R_2 \rightarrow R_1$$

$$\begin{array}{ccccc}
 1 & 0 & 0 & 0 & -2 \\
 0 & 1 & 0 & 0 & 4 \\
 0 & 0 & 1 & 0 & -1 \\
 0 & 0 & 0 & 1 & -4
 \end{array}$$

Return the matrix to equation form:

$$\begin{array}{rcl}
 a & & = -2 \\
 & b & = 4 \\
 & & c & = -1 \\
 & & & d & = -4
 \end{array}$$

$$a = -2, b = 4, c = -1, d = -4$$

Note: the system could also be solved using Gaussian elimination to put the matrix in row echelon form, using back substitution to obtain the final solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2b_1 + 5b_3 - 3b_2 \\ 0 & 1 & 0 & b_1 - 2b_3 + b_2 \\ 0 & 0 & -3 & -3b_1 + 7b_3 - 5b_2 \end{array} \right] \xrightarrow{1/3R_3 \rightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2b_1 + 5b_3 - 3b_2 \\ 0 & 1 & 0 & b_1 - 2b_3 + b_2 \\ 0 & 0 & 1 & b_1 - 7/3b_3 + 5/3b_2 \end{array} \right] \xrightarrow{R_1 + R_3 \rightarrow R_1} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -b_1 + 8/3b_3 - 4/3b_2 \\ 0 & 1 & 0 & b_1 - 2b_3 + b_2 \\ 0 & 0 & 1 & b_1 - 7/3b_3 + 5/3b_2 \end{array} \right]$$

Our equations are then:

$$\begin{aligned} x_1 &= -b_1 + 8/3b_3 - 3b_2 \\ x_2 &= b_1 - 2b_3 + b_2 \\ x_3 &= b_1 - 7/3b_3 + 5/3b_2 \end{aligned}$$

We take the b_1 , b_2 , and b_3 values from problem A and plug them in. We get this:

A) $b_1=1 \quad b_2=6 \quad b_3=3$

$$\begin{aligned} x_1 &= -1 + 8/3(3) - 3(6) = -1 + 8 - 18 = -11 \\ x_2 &= 1 - 2(3) + (6) = 1 - 6 + 6 = 1 \\ x_3 &= 1 - 7/3(3) + 5/3(6) = 1 - 7 + 10 = 3 \end{aligned}$$

$$\begin{aligned} x_1 &= -11 \\ x_2 &= 1 \\ x_3 &= 3 \end{aligned}$$

B) $b_1=-2 \quad b_2=1 \quad b_3=9$

$$\begin{aligned} x_1 &= 2 + 8/3(9) - 3(1) = -1 + 24 - 3 = 20 \\ x_2 &= -2 - 2(9) + (1) = -2 - 18 + 1 = -19 \\ x_3 &= -2 - 7/3(9) + 5/3(1) = -2 - 21 + 5/3 = -21 \frac{1}{3} \end{aligned}$$

$$\begin{aligned} x_1 &= 20 \\ x_2 &= -19 \\ x_3 &= -21 \frac{1}{3} \end{aligned}$$

Section 2.1-2.2

Consider the invertible matrix

$$A = \begin{bmatrix} 1 & 0 & 8 \\ 6 & 5 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

(A) Find the determinant of A using cofactor expansion along the first row

(B) Find the inverse of A by finding its adjoint and using the formula:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

Solution

(A)

The determinant of matrix A along the first row can be found by using the following equation:

$$\det(A) = A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13}$$

where C_{ij} is the cofactor of entry A_{ij} .

The cofactors of A_{11} , A_{12} , and A_{13} are:

$$C_{11} = (-1)^{1+1} * \begin{vmatrix} 5 & 2 \\ 1 & 5 \end{vmatrix} \Rightarrow (-1)^2 * \begin{vmatrix} 5 & 2 \\ 1 & 5 \end{vmatrix} \Rightarrow 1 * (5*5 - 2*1) = 23$$

$$C_{12} = (-1)^{1+2} * \begin{vmatrix} 6 & 2 \\ 2 & 5 \end{vmatrix} \Rightarrow (-1)^3 * \begin{vmatrix} 6 & 2 \\ 2 & 5 \end{vmatrix} \Rightarrow -1 * (6*5 - 2*2) = -26$$

$$C_{13} = (-1)^{1+3} * \begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix} \Rightarrow (-1)^4 * \begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix} \Rightarrow 1 * (6*1 - 5*2) = -4$$

Therefore by substituting the cofactor values into our equation we get:

$$\det(A) = A_{11}*23 + A_{12}*-26 + A_{13}*-4$$

We can obtain the values for A_{ij} by inspection from matrix A to obtain

$$\det(A) = 1*23 + 0*-26 + 8*-4$$

Therefore: $\det(A) = -9$

(B)

The adjoint of A is the transpose of the matrix of cofactors from A, and can be modeled by the following matrix:

Given the matrices $A = \begin{vmatrix} 5 & 3 & 1 \\ -1 & -7 & 3 \\ 4 & 1 & 2 \end{vmatrix}$ and $B = \begin{vmatrix} 8 & 0 & -3 \\ 4 & 2 & 1 \\ -2 & 1 & 3 \end{vmatrix}$

- Show that both matrices are invertible.
- Find the determinant of A^{-1} and B^{-1} .
- Numerically show $\det(AB) = \det(A) \cdot \det(B)$.
- Prove $\det(A^{-1}B^{-1}) = \det(A^{-1}) \cdot \det(B^{-1})$.
- Find and list the number of inversions in each row of A and B. Tell whether each is even or odd.

Notation:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

The figure above represents a matrix, not the determinant of one.

prove, or show for this specific example
inversions are associated with permutations, not rows/entries of matrix

Answers:

Part a.)

Both matrices are invertible if $\det(A) \neq 0$ and $\det(B) \neq 0$. Therefore if we find the determinant of A and the determinant of B and they aren't zero, we know that they are invertible because they are equivalent statements.

Matrix A: Cofactor expansion of Row 3

$$\det(A) = 4 \begin{vmatrix} 3 & 1 \\ -7 & 3 \end{vmatrix} - \begin{vmatrix} 5 & 1 \\ 1 & 3 \end{vmatrix} + 2 \begin{vmatrix} 5 & 3 \\ -1 & -7 \end{vmatrix}$$
$$= 4(9+7) - (15+1) + 2(-35+3) = (4)(16) - 16 + (2)(32) = -16$$
$$\det(A) = -16$$

Because $\det(A) \neq 0$ we know that Matrix A is invertible.

Matrix B: Cofactor expansion of Row 1

$$\det(B) = 8 \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} - 0 - 3 \begin{vmatrix} 4 & 2 \\ -2 & 1 \end{vmatrix}$$
$$= 8(6-1) - 3(4+4) = (8)(5) - (3)(8) = 16$$
$$\det(B) = 16$$

Because $\det(B) \neq 0$ we know that Matrix B is invertible.

Part b.)

$$\det(A^{-1}) = 1/\det(A) = -1/16$$

$$\det(B^{-1}) = 1/\det(B) = 1/16$$

Part c.)

AB =

$$\begin{vmatrix} 50 & 7 & -9 \\ -42 & -11 & 5 \\ 32 & 4 & -5 \end{vmatrix}$$

Matrix AB: Cofactor Expansion along Row 1

$$\det(AB) = 50 \begin{vmatrix} -11 & 5 \\ 4 & -5 \end{vmatrix} - 9 \begin{vmatrix} -42 & 5 \\ 32 & -5 \end{vmatrix} - 42 \begin{vmatrix} -42 & -11 \\ 32 & 4 \end{vmatrix}$$
$$= 50(55-20) - 9(210-160) - 42(-168+352) = (50)(35) - (9)(50) - (9)(184) = -256$$
$$\det(AB) = -256$$

$$\det(A) \times \det(B) \stackrel{?}{=} \det(AB)$$

$$(-16)(16) \stackrel{?}{=} -256$$

$-256 \stackrel{?}{=} -256$ Yes, the determinant of A multiplied by the determinant of B is the same as the determinant of the matrix AB, which is the product of the matrix A and matrix B.

Part d.)

$$\det(A^{-1}B^{-1}) \stackrel{?}{=} \det(A^{-1}) \times \det(B^{-1})$$

$$\det(A^{-1}B^{-1}) = [\det(A) \times \det(B)]^{-1} = \det(A)^{-1} \times \det(B)^{-1} = \det(A^{-1}) \times \det(B^{-1})$$

$$\det(A^{-1}) \times \det(B^{-1}) \stackrel{?}{=} \det(A^{-1}) \times \det(B^{-1}) \text{ Which shows that the original statement is true.}$$

Part e.)

Matrix A

Row 1: (5,3) (5,1) (3,1) 3 inversions Odd

Row 2: (-1,-7) 1 inversion Odd

Row 3: (4,1) (4,2) 2 inversions Even

Matrix B

Row 1: (8,0) (8,-3) (0,-3) 3 inversions Odd

Row 2: (4,2) (4,1) 2 inversions Even

Row 3: none 0 inversions Even

completely misapplied

1. Solve the following set of equations using Gauss-Jordan Elimination (30 Pts)

$$\begin{aligned} 6x_1 + 6x_2 - 3x_3 &+ 3x_5 = 0 \\ -3x_1 - 3x_2 + 6x_3 - 3x_4 + 3x_5 &= 0 \\ 3x_1 + 3x_2 - 6x_3 &+ 3x_5 = 0 \\ 3x_3 + 3x_4 + 3x_5 &= 0 \end{aligned}$$

Answer:

First form the equations into an augmented matrix

From here perform elementary row operations on the matrix to get it in reduced row echelon form.

$$\left| \begin{array}{cccccc} 6 & 6 & -3 & 0 & 3 & 0 \\ -3 & -3 & 6 & -3 & 3 & 0 \\ 3 & 3 & -6 & 0 & -3 & 0 \\ 0 & 0 & 3 & 3 & 3 & 0 \end{array} \right|$$

$$\begin{aligned} R_1 &\longleftrightarrow R_2 \\ 2R_1 + R_2 &\longrightarrow R_3 \\ R_1 + R_3 &\longrightarrow R_3 \end{aligned}$$

$$\left| \begin{array}{cccccc} -3 & -3 & 6 & -3 & 3 & 0 \\ 0 & 0 & 9 & -6 & 9 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 3 & 3 & 0 \end{array} \right|$$

$$\begin{aligned} -\frac{1}{3} R_1 &\longrightarrow R_1 \\ -3R_4 + R_2 &\longrightarrow R_2 \\ -\frac{1}{3} R_3 &\longrightarrow R_3 \\ \frac{1}{3} R_4 &\longrightarrow R_4 \end{aligned}$$

$$\left| \begin{array}{cccccc} 1 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & -15 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right|$$

$$\begin{aligned} R_2 &\longleftrightarrow R_4 \\ 15R_3 + R_4 &\longrightarrow R_4 \\ -R_3 + R_2 &\longrightarrow R_2 \\ -R_3 + R_1 &\longrightarrow R_1 \\ 2R_2 + R_1 &\longrightarrow R_1 \end{aligned}$$

$$\left| \begin{array}{cccccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right|$$

Change the augmented matrix back to a system of equations

$$\begin{aligned} x_1 + x_2 &+ x_5 = 0 \\ x_3 &+ x_5 = 0 \\ x_4 &= 0 \end{aligned}$$

We set x_2 equal to t and x_5 equal to s to give the following set of solutions.

two-parameter family

$$x_1 = -t - s, \quad x_2 = s, \quad x_3 = -t, \quad x_4 = 0, \quad x_5 = t$$

(20 points for finding the reduced matrix and 10 points awarded for the correct solutions.)

2. Consider the matrices:

(55 Points)

$$A = \begin{vmatrix} 2 & 1 & -2 \\ 0 & -3 & 0 \end{vmatrix}$$

B =

$$\begin{vmatrix} 3 & 5 \\ 1 & -4 \\ 3 & 0 \end{vmatrix}$$

C =

$$\begin{vmatrix} 12 & 4 & 7/2 \\ -3 & -9 & 8 \end{vmatrix}$$

Compute the following:

i. $4A - 2C$

ii. AB

iii. $(AB)^T$

iv. $(AB)^{-1}$ (using row operations)

Answers:

$$\text{i. } 4A = \begin{vmatrix} 8 & 4 & -8 \\ 0 & -12 & 0 \end{vmatrix}$$

$$-2C = \begin{vmatrix} -24 & -8 & -7 \\ 6 & 18 & -16 \end{vmatrix}$$

$$\text{thus } 4A + (-2C) = \begin{vmatrix} -16 & -4 & -15 \\ 6 & 6 & -16 \end{vmatrix}$$

$$\text{ii. } AB = \begin{vmatrix} 2*3+1*1+2*3 & 2*5+1*-4+2*0 \\ 0*3+-3*1+0*3 & 0*5+-3*-4+0*0 \end{vmatrix} = \begin{vmatrix} 1 & 6 \\ -3 & -12 \end{vmatrix}$$

$$\text{iii. } (AB)^T = \begin{vmatrix} 1 & -3 \\ 6 & -12 \end{vmatrix}$$

iv. To begin, adjoin the identity matrix to the right side of AB. Whatever row operations are applied to matrix AB until it is reduced to I, perform the same on the right side to achieve $(AB)^{-1}$.

$$\begin{vmatrix} 1 & 6 \\ -3 & 12 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \xrightarrow{3R_1 + R_2 \rightarrow R_2} \begin{vmatrix} 1 & 6 \\ 0 & 30 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \xrightarrow{1/30R_2 \rightarrow R_2}$$

$$\begin{vmatrix} 1 & 6 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 1/10 & 1/30 \end{vmatrix} \xrightarrow{-6R_2 + R_1 \rightarrow R_1} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 2/5 & -1/5 \\ 1/10 & 1/30 \end{vmatrix}$$

We have obtained the identity matrix on the left. Thus,

$$(AB)^{-1} = \begin{vmatrix} 2/5 & -1/5 \\ 1/10 & 1/30 \end{vmatrix}$$

(12 points for part i, ii, and iii. 19 points for part iv.)

3. Find the determinant of the matrix A using three different methods we learned.
(45 points)

$$A = \begin{vmatrix} 0 & 0 & 3 \\ 1 & 4 & 2 \\ 3 & 6 & 9 \end{vmatrix}$$

Answer:

Method 1 involves the summation of all signed elementary products from A.

$$\det(A) =$$

$$\begin{vmatrix} 0 & 0 & 3 \\ 1 & 4 & 2 \\ 3 & 6 & 9 \end{vmatrix}$$

$$\begin{aligned} & (0 \cdot 4 \cdot 9 + 0 \cdot 2 \cdot 3 + 3 \cdot 1 \cdot 6 - 3 \cdot 4 \cdot 3 - 0 \cdot 2 \cdot 6 - 0 \cdot 1 \cdot 9) \\ &= (0 + 0 + 18 - 36 - 0 - 0) = 18 - 36 \\ &= -18 = \det(A). \end{aligned}$$

Method 2 uses row reduction to evaluate a determinant.

$$\det(A) =$$

$$\begin{vmatrix} 0 & 0 & 3 \\ 1 & 4 & 2 \\ 3 & 6 & 9 \end{vmatrix}$$

Divide row 3
by three.

$$= (3)$$

$$\begin{vmatrix} 0 & 0 & 3 \\ 1 & 4 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$-R_3 + R_2 \longrightarrow R_2$$

$$= (3)(-1) \cdot A$$

$$\begin{vmatrix} 0 & 0 & 3 \\ 0 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix}$$

this is not a Δ matrix!
 $\det(A)$ = the product of the entries along the main diagonal of the matrix because $A_{3 \times 3}$ is a square lower triangular matrix.

$$\det(A) = (3)(-1)(1)(2)(3) = -18.$$

Method 3 exemplifies cofactor expansion along the first row.

$$\det(A) =$$

$$\begin{vmatrix} 0 & 0 & 3 \\ 1 & 4 & 2 \\ 3 & 6 & 9 \end{vmatrix}$$

$$= 3 \cdot$$

$$\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix}$$

notation?

Because the first two entries in row 1 of matrix A are 0s, when multiplied by their cofactor, they will produce a product of 0, leaving only 3 times its cofactor.

$$= 3(1 \cdot 6 - 4 \cdot 3) = \det(A) = -18.$$

(15 points for each method that reaches the correct solution.)

4. Prove Theorem 1.7.3, which states:

If A is an invertible symmetric matrix, then A^{-1} is symmetric. (20 Points)

Proof:

Definition of a symmetric matrix for a matrix A is $A = A^T$

We can use theorem 1.4.10 which says if A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

Good

Since A is an invertible matrix and symmetric, then using theorem 1.4.10 and the definition of symmetric, we know $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ and so A^{-1} is symmetric.

1. Given matrices:

$$A = \begin{bmatrix} 5 & 9 & 2 \\ 1 & 6 & 8 \\ 0 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 10 & 8 \\ 7 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 4 \\ 9 & 1 \\ 3 & 2 \end{bmatrix}$$

a. Find $AB + C$

$$\begin{bmatrix} 5 & 9 & 2 \\ 1 & 6 & 8 \\ 0 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 10 & 8 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 119 & 85 \\ 119 & 81 \\ 31 & 20 \end{bmatrix} = AB$$

$$\begin{bmatrix} 119 & 85 \\ 119 & 81 \\ 31 & 20 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 9 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 121 & 89 \\ 128 & 82 \\ 34 & 22 \end{bmatrix} = AB + C$$

b. Find $A(B+C)$

First, find $B+C$.

$$\begin{bmatrix} 3 & 1 \\ 10 & 8 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 9 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 19 & 9 \\ 10 & 6 \end{bmatrix}$$

Then, multiply A on the left side of $B+C$

$$\begin{bmatrix} 5 & 9 & 2 \\ 1 & 6 & 8 \\ 0 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 5 & 5 \\ 19 & 9 \\ 10 & 6 \end{bmatrix} = \begin{bmatrix} 216 & 118 \\ 199 & 107 \\ 49 & 27 \end{bmatrix} = A(B+C)$$

c. Find $(B+C)A$

There is no solution because you cannot multiply a 3×2 matrix by a 3×3 matrix.

2. Use Gauss-Jordan Elimination to find the inverse of

$$\begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array} \right] R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array} \right] -3R_1 + R_2 \rightarrow R_2, -2R_1 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array} \right] R_3 - R_2 \rightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \end{array} \right] R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 4 & -10 & 1 & -3 & 0 \end{array} \right] -4R_2 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{array} \right] -\frac{1}{10}R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array} \right] -3R_3 + R_1 \rightarrow R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array} \right]$$

3. If A is an invertible matrix, show that AA^T and A^TA are also invertible. Hint: show first that $A = A^T$ — *not true for all invertible matrices!*

Solution:

First, we show that if A is invertible, then so is A^T .

$$A^T(A^{-1})^T = (A^{-1})^T(A^T) = I$$

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(A^{-1})^T(A^T) = (AA^{-1})^T = I^T = I$$

We just showed that since A is invertible so is A^T . Both AA^T and A^TA are products of invertible matrices and therefore themselves invertible.

4. Given the following matrix:

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{bmatrix}$$

a. Find the determinant using any method.

Solution:

$$\det \begin{bmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{bmatrix}$$

$$\begin{aligned} &= [(2 \times 0 \times 1) + (3 \times 1 \times 2) + (-1 \times 7 \times -5)] - [(-1 \times 0 \times 2) + (1 \times -3 \times 1) + (2 \times 3 \times 7)] \\ &= [0 + 6 + 35] - [0 + (-5) + 42] \\ &= 41 - 37 \\ &= 4 \end{aligned}$$

b. Find the minors for the matrix

Solution:

$$M_{11} = -1 \begin{bmatrix} 0 & -5 \\ 7 & 2 \end{bmatrix} = -35$$

$$M_{12} = 1 \begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix} = 11$$

$$M_{13} = 2 \begin{bmatrix} 3 & 0 \\ 1 & 7 \end{bmatrix} = 42$$

$$M_{21} = 3 \begin{bmatrix} 1 & 2 \\ 7 & 2 \end{bmatrix} = -36$$

$$M_{22} = 0 \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} = 0$$

$$M_{23} = -5 \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix} = 40$$

$$M_{31} = 1 \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = -5$$

$$M_{32} = 7 \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix} = -7$$

$$M_{33} = 2 \begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix} = -6$$

c. Find the Cofactor matrix.

Solution:

$$\begin{bmatrix} -35 & -11 & 42 \\ 36 & 0 & -40 \\ -5 & 7 & -6 \end{bmatrix}$$

(9)

Test Question – 1.1 to 1.3
Worth 30 Pts.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 4 & 2 \\ 0 & 6 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & -1 \\ -6 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

Question: Using matrices A and B, determine the following where possible.

- A. $A + B$ B. $A * B$ C. $B^T + 3A^T$ D. $-(B^T * A)$

Solution:

A. Worth 7 pts.

$$\begin{bmatrix} 1 & 3 & 5 \\ -1 & 4 & 2 \\ 0 & 6 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -1 \\ -6 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 4 \\ -7 & 6 & 3 \\ 1 & 7 & 3 \end{bmatrix}$$

Each corresponding row and column value added together. For example row 1, column 1 of A added with row 1, column 1 of B. This value is put into row 1, column 1 of the solution. $(1+2=3)$ This is done for each corresponding row and column in A and B.

B. Worth 7 Pts.

$$\begin{bmatrix} 1 & 3 & 5 \\ -1 & 4 & 2 \\ 0 & 6 & -1 \end{bmatrix} * \begin{bmatrix} 2 & 1 & -1 \\ -6 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1) - 6(3) + 5(1) & 1(1) + 2(3) + 1(5) & -1(1) + 1(3) + 4(5) \\ 2(-1) - 6(4) + 1(2) & 1(-1) + 2(4) + 1(2) & -1(-1) + 1(4) + 4(2) \\ 2(0) - 6(6) + 1(-1) & 1(0) + 2(6) + 1(-1) & -1(0) + 1(6) + 1(2) \end{bmatrix}$$

To solve for this matrix problem, values going across the rows in matrix A are multiplied with values going down the column in matrix B then each value is added together to get a sum for each possible solution. For example, Matrix A Row 1 * Matrix B Col 1 goes in C11. Matrix A Row 2 * Matrix B Column 3 goes in C23.

$$= \begin{bmatrix} 2 - 18 + 5 & 1 + 6 + 5 & -1 + 3 + 20 \\ -2 - 24 + 2 & -1 + 8 + 2 & 1 + 4 + 8 \\ 0 - 36 - 1 & 0 + 12 - 1 & 0 + 6 - 4 \end{bmatrix}$$

no antecedant

$$= \begin{bmatrix} -11 & 12 & 22 \\ -24 & 9 & 13 \\ -37 & 11 & 2 \end{bmatrix}$$

C. Worth 8 Pts.

$$B^T = \begin{bmatrix} 2 & 1 & -1 \\ -6 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & -6 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

The transverse is found by switching the column number with the row number. For example, the value in C23 would be placed in C32.

$$3A^T = 3 \begin{bmatrix} 1 & 3 & 5 \\ -1 & 4 & 2 \\ 0 & 6 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 4 & 6 \\ 5 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 \\ 9 & 12 & 18 \\ 15 & 6 & -3 \end{bmatrix}$$

Transverse of A was taken and each value inside was multiplied by 3.

$$B^T + 3A^T = \begin{bmatrix} 2 & -6 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 0 \\ 9 & 12 & 18 \\ 15 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 5 & -9 & 1 \\ 10 & 14 & 19 \\ 15 & 7 & 1 \end{bmatrix}$$

The two matrices were added together like in part A.

D. Worth 8 Pts.

$$B^T = \begin{bmatrix} 2 & 1 & -1 \\ -6 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & -6 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

$$-(B^T A) = -1 \begin{bmatrix} 2 & -6 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 3 & 5 \\ -1 & 4 & 2 \\ 0 & 6 & -1 \end{bmatrix}$$

These matrices were multiplied together similar to part B.

$$= -1 \begin{bmatrix} 1(2) - 1(-6) + 0(1) & 3(2) + 4(-6) + 6(1) & 5(2) + 2(-6) - 1(1) \\ 1(1) - 1(2) + 0(1) & 3(1) + 4(2) + 6(1) & 5(1) + 2(2) - 1(1) \\ 1(-1) - 1(1) + 0(4) & 3(1) + 4(1) + 6(4) & 5(-1) + 2(1) - 1(4) \end{bmatrix}$$

$$= -1 \begin{bmatrix} 2+6+0 & 6-24+6 & 10-12-1 \\ 1-2+0 & 3+8+6 & 5+4-1 \\ -1-1+0 & -3+4+24 & -5+2-4 \end{bmatrix}$$

$$= -1 \begin{bmatrix} 8 & -12 & -3 \\ -1 & 17 & 8 \\ -2 & 25 & -7 \end{bmatrix}$$

-1 was multiplied to each value inside the matrix similar to what was done in part C.

$$= \begin{bmatrix} -8 & 12 & 3 \\ 1 & -17 & -8 \\ 2 & -25 & 7 \end{bmatrix}$$

2. Find the inverse of the following matrices

a. $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 3 & 5 \\ 3 & 10 & 6 \\ 2 & 6 & 5 \end{bmatrix}$

Solution.

For all the matrices, you set up a double matrix in the form $[A | I]$, then convert to the form $[I | A^{-1}]$. For the sake of simplicity, in my notation, commas denote "then," rather than interchangeable EROs.

a. $\begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 2 & 5 & | & 0 & 1 \end{bmatrix}$ $R_2 - 2R_1 \rightarrow R_2$, then $R_1 - 2R_2 \rightarrow R_1$ $\begin{bmatrix} 1 & 0 & | & 5 & -2 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$

The inverse is $\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

Another possible, equally correct answer would be using the 2*2 shortcut

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\frac{1}{5-4} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

10 pts for either solution

b. $\begin{bmatrix} 2 & 3 & 5 & | & 1 & 0 & 0 \\ 3 & 10 & 6 & | & 0 & 1 & 0 \\ 2 & 6 & 5 & | & 0 & 0 & 1 \end{bmatrix}$ $R_1 \leftrightarrow R_2$, new $R_1 - R_3 \rightarrow R_1$, $R_3 - \text{new } R_2 \rightarrow R_3$, $R_3 \leftrightarrow R_2$

$\begin{bmatrix} 1 & 4 & 1 & | & 0 & 1 & -1 \\ 0 & 1 & 3 & | & -1 & -2 & 4 \\ 0 & 3 & 0 & | & -1 & 0 & 1 \end{bmatrix}$ 10 pts (or similar matrix) $R_3 - 3R_2 \rightarrow R_3$, $-1/9R_3$

$\begin{bmatrix} 1 & 4 & 1 & | & 0 & 1 & -1 \\ 0 & 1 & 3 & | & -1 & -2 & 4 \\ 0 & 0 & 1 & | & -2/9 & -6/9 & 11/9 \end{bmatrix}$ $R_2 - 3R_3 \rightarrow R_2$, $R_1 - R_3 \rightarrow R_1$

$\begin{bmatrix} 1 & 4 & 0 & | & 2/9 & 5/3 & -20/9 \\ 0 & 1 & 0 & | & -1/3 & 0 & 1/3 \\ 0 & 0 & 1 & | & -2/9 & -6/9 & 11/9 \end{bmatrix}$ $R_1 - 4R_2 \rightarrow R_1$

$\begin{bmatrix} 1 & 0 & 0 & | & 14/9 & 5/3 & -32/9 \\ 0 & 1 & 0 & | & -1/3 & 0 & 1/3 \\ 0 & 0 & 1 & | & -2/9 & -2/3 & 11/9 \end{bmatrix}$

This inverse is $\begin{bmatrix} 14/9 & 5/3 & -32/9 \\ -1/3 & 0 & 1/3 \\ -2/9 & -6/9 & 11/9 \end{bmatrix}$ 30 pts if solution 100% correct

$[-1/3 \ 0 \ 1/3]$ Minus 3 pts if 1 entry wrong in any way

$[-2/9 \ -6/9 \ 11/9]$ Minus 8 if 1 row wrong. Total = /40

(Of course, in this procedure, there are almost an infinite # of ERO combos that would get the same answer. However, the end result will always be the same)

Solve the system using the identity $A^{-1}b=x$.

$$x_1 + 3x_2 + x_3 = 4$$

$$2x_1 + 2x_2 + x_3 = -1$$

$$2x_1 + 3x_2 + 1x_3 = 3$$

a) $x_1=7$ $x_2=4$ $x_3=17$

b) $x_1=17$ $x_2=4$ $x_3=117$

c) $x_1=7$ $x_2=7$ $x_3=17$

d) $x_1=7$ $x_2=4$ $x_3=4$

e) $x_1=4$ $x_2=7$ $x_3=17$

what are these?

Answer: a)

First we need to find inverse of the coefficient matrix.

$$A = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array} \right]$$

Through the following Elementary Row Operations the inverse is found:

a) Take the 3rd row, subtract the 2nd, insert in the 2nd.

Take the -2 times the 1st row, add the 3rd row and insert into 3rd.

b) Take 3 times the 2nd row, add the 3rd row, insert into 3rd row.

c) Take -3 times the 2nd row, add the 1st row, insert into the 1st row.

d) Take -1 times the 3rd row, add the 1st row, insert into the first row.

This gives you A^{-1} :

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array} \right]$$

$$b = \left[\begin{array}{c} 4 \\ -1 \\ 3 \end{array} \right]$$

Therefore $A^{-1}b =$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array} \right] \times \left[\begin{array}{c} 4 \\ -1 \\ 3 \end{array} \right] = \left[\begin{array}{c} 7 \\ 4 \\ 17 \end{array} \right]$$

And $x_1=7$ $x_2=4$ $x_3=17$

30 POINTS

Problem:

For following matrix first find signed elementary product (S.E.P) associated to p ; if $p(1)=3$ $p(2)=1$ $p(3)=2$. Then find $\det(A)$ using cofactor expansion. Afterwards explain the difference for formulating the signs of a signed elementary product versus using cofactor expansion.

$$A = \begin{bmatrix} x & y & z \\ a & b & c \\ j & k & l \end{bmatrix}$$

Solution:

S.E.P if $p(1)=3$ $p(2)=1$ $p(3)=2$. (worth 15 points)

$$\text{Then } a_{1p(1)} * a_{2p(2)} * a_{3p(3)}$$

$$= a_{13} * a_{21} * a_{32}$$

$$= zak \text{ (2 inversions + sign)}$$

$\det(A)$ using cofactor expansion (along row 1): (worth 25 points)

$$A = \begin{bmatrix} x & y & z \\ a & b & c \\ j & k & l \end{bmatrix}$$

$$\det(A) = x * \det \begin{bmatrix} b & c \\ k & l \end{bmatrix} - y * \det \begin{bmatrix} a & c \\ j & l \end{bmatrix} + z * \det \begin{bmatrix} a & b \\ j & k \end{bmatrix}$$

$$= x(bl - ck) - y(al - cj) + z(ak - bj)$$

$$= xbl - xck - yal + ycj + zak - zbj$$

(notice signs of zak for both S.E.P. and cofactor expansion are same.)

Signs: (Worth 10 points)

For S.E.P. count number of inversions if number of inversions is even then multiply E.P. by 1 if odd number of inversions then multiply E.P. by (-1) .

For Cofactor Expansion sign in front of $a_{ij}M_{ij}$ is found by $(-1)^{i+j}$.

Test Questions

1. Evaluate the determinant of the following matrix via cofactor expansion.

$$A = \begin{bmatrix} 0 & -1 & 3 & 5 \\ 5 & -4 & -6 & 3 \\ 0 & 2 & 10 & 0 \\ 0 & 1 & 7 & 0 \end{bmatrix} \quad \det(A) = ?$$

Key:

$$\det(A) = \begin{vmatrix} 0 & -1 & 3 & 5 \\ 5 & -4 & -6 & 3 \\ 0 & 2 & 10 & 0 \\ 0 & 1 & 7 & 0 \end{vmatrix}$$

$$\det(A) = -5 \begin{vmatrix} -1 & 3 & 5 \\ 2 & 10 & 0 \\ 1 & 7 & 0 \end{vmatrix}$$

$$\det(A) = -5 \cdot 5 \begin{vmatrix} 2 & 10 \\ 1 & 7 \end{vmatrix}$$

$$\det(A) = -5 \cdot 5 (14 - 10)$$

$$\text{Answer : } \det(A) = -100$$

2. Given $p(x) = x^2 - 2x + 3$ and $A = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$ *Find solve for $P(A^T)$*

Key:

$$A^T = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$$

$$p(A^T) = (A^T)^2 - 2A^T + 3I = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}^2 - 2 \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$p(A^T) = \begin{bmatrix} 6 & 3 \\ 2 & 11 \end{bmatrix}$$

3. $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$

Find A^T multiplied by A .

Is the answer a symmetric matrix? *— always true?*

Key:

$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} =$$

$$\begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix} \text{ The answer is symmetric}$$

4. Prove $(AB)^{-1} = B^{-1}A^{-1}$

Key:

if $(AB)^{-1} \cdot B^{-1}A^{-1} = B^{-1}A^{-1}(AB) = I$
then $(AB)^{-1} = B^{-1}A^{-1}$

Part 1:

$$B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B \quad \text{associative law}$$

$$B^{-1}(A^{-1}A)B = B^{-1}(I)B \quad \text{Def. Invert. matrix}$$

$= B^{-1}(IB) \quad \text{assoc.}$

$$B^{-1}(I)B = B^{-1}B \quad \text{prop. of identity}$$

$$B^{-1}B = I \quad \text{Def. Invert. matrix}$$

Part 2:

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} \quad \text{associative law}$$

$$A(BB^{-1})A^{-1} = A(I)A^{-1} \quad \text{Def. Invert. matrix}$$

$= A(A^{-1}A) \quad \text{assoc.}$

$$A(I)A^{-1} = AA^{-1} \quad \text{prop. of identity}$$

$$AA^{-1} = I \quad \text{Def. Invert. matrix}$$

Therefore: $(AB)^{-1} = B^{-1}A^{-1}$

Math 343 Sample Exam 1 Problems

1. Find x such that $Ax = B$, using $x = A^{-1}b$, when $A|b =$

$$\begin{array}{ccc|ccc} -1 & -4 & 2 & 1 & -32 & \\ 2 & -1 & 7 & 9 & 14 & \\ -1 & 1 & 3 & 1 & 11 & \\ 1 & -2 & 1 & -4 & -4 & \end{array} :$$

47
good

$$A^{-1}: \begin{array}{ccc|ccc} -1 & -4 & 2 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 7 & 9 & 0 & 1 & 0 & 0 \\ -1 & 1 & 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & -4 & 0 & 0 & 0 & 1 \end{array} \quad \begin{array}{l} \text{Multiply R1 by -1 and put in R1, add 2 (R1) plus} \\ \text{R2 and put in R2, subtract R1 from R3 and put in} \\ \text{R3, and add R1 + R4 and put in R4.} \end{array}$$

$$A^{-1}: \begin{array}{ccc|ccc} 1 & 4 & -2 & -1 & -1 & 0 & 0 & 0 \\ 0 & -9 & 11 & 11 & 2 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -6 & 3 & -3 & 1 & 0 & 0 & 1 \end{array} \quad \begin{array}{l} \text{Add 5 (R2) plus 9 (R3) and put in R3 and add} \\ \text{6 (R3) plus 5 (R4) and put in R4.} \end{array}$$

$$A^{-1}: \begin{array}{ccc|ccc} 1 & 4 & -2 & -1 & -1 & 0 & 0 & 0 \\ 0 & -9 & 11 & 11 & 2 & 1 & 0 & 0 \\ 0 & 0 & 64 & 55 & 1 & 5 & 9 & 0 \\ 0 & 0 & 21 & -15 & -1 & 0 & 6 & 5 \end{array} \quad \begin{array}{l} \text{Subtract 21 (R3) from 64 (R4) and put in R4.} \end{array}$$

$$A^{-1}: \begin{array}{ccc|ccc} 1 & 4 & -2 & -1 & -1 & 0 & 0 & 0 \\ 0 & -9 & 11 & 11 & 2 & 1 & 0 & 0 \\ 0 & 0 & 64 & 55 & 1 & 5 & 9 & 0 \\ 0 & 0 & 0 & -2115 & -85 & -105 & 195 & 320 \end{array} \quad \begin{array}{l} \text{Subtract 2115 (R1) from R4 and put} \\ \text{in R1, add 2115 (R2) plus} \\ \text{11 (R4) and put in R2, and add} \\ \text{423 (R3) plus 11 (R4) and put in} \\ \text{R3.} \end{array}$$

$$A^{-1}: \begin{array}{ccc|ccc} -2115 & -8460 & 4230 & 0 & 2030 & -105 & 195 & 320 \\ 0 & -19035 & 23265 & 0 & 3295 & 960 & 2145 & 3520 \\ 0 & 0 & 27072 & 0 & -512 & 960 & 5952 & 3520 \\ 0 & 0 & 0 & -2115 & -85 & -105 & 195 & 320 \end{array} \quad \begin{array}{l} \text{Subtract 11 (R1)} \\ \text{plus 2 (R2) and} \\ \text{put in R1 and} \\ \text{subtract 64 (R2)} \\ \text{from 55 (R3)} \\ \text{and put in R2.} \end{array}$$

$$A^{-1}: \begin{array}{ccc|ccc} 23265 & 54990 & 0 & 0 & -15740 & 3075 & 2145 & 3520 \\ 0 & 1218240 & 0 & 0 & -239040 & -8640 & 190080 & -31680 \\ 0 & 0 & 27072 & 0 & -512 & 960 & 5952 & 3520 \\ 0 & 0 & 0 & -2115 & -85 & -105 & 195 & 320 \end{array} \quad \begin{array}{l} \text{Subtract 13 (R2)} \\ \text{from 288 (R1) and} \\ \text{put in R1.} \end{array}$$

$$A^{-1}: \left| \begin{array}{cccc|cccc} 6700320 & 0 & 0 & 0 & -1425600 & 997920 & -1853280 & 1425600 \\ 0 & 1218240 & 0 & 0 & -239040 & -8640 & 190080 & -31680 \\ 0 & 0 & 27072 & 0 & -512 & 960 & 5952 & 3520 \\ 0 & 0 & 0 & -2115 & -85 & -105 & 195 & 320 \end{array} \right|$$

Divide R1 by 6700320 and put in R1, divide R2 by 1218240 and put in R2, divide R3 by 27072 and put in R3, and divide R4 by -2115 and put in R4.

$$A^{-1}: \left| \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -10/47 & 7/47 & -13/47 & 10/47 \\ 0 & 1 & 0 & 0 & -83/423 & -1/141 & 22/141 & -11/423 \\ 0 & 0 & 1 & 0 & -8/423 & 5/141 & 31/141 & 55/423 \\ 0 & 0 & 0 & 1 & 17/423 & 7/141 & -13/141 & -64/423 \end{array} \right|$$

$$x = A^{-1}b = \left| \begin{array}{cccc|c|c} -10/47 & 7/47 & -13/47 & 10/47 & -32 & 5 \\ -83/423 & -1/141 & 22/141 & -11/423 & 14 & 8 \\ -8/423 & 5/141 & 31/141 & 55/423 & 11 & 3 \\ 17/423 & 7/141 & -13/141 & -64/423 & -4 & -1 \end{array} \right| =$$

2. Show that if $AB = 0$ and A is invertible, then $B = 0$:

$$AB = 0$$

$$A^{-1}(AB) = A^{-1}0$$

$$(A^{-1}A)B = 0$$

$$IB = 0$$

$$B = 0$$

properties / justifications?

3. Find the $\det(A)$, $A = \begin{vmatrix} 0 & 1 & 5 & 2 \\ 0 & 2 & 0 & 4 \\ 3 & 3 & 7 & 0 \\ 0 & 4 & 3 & 1 \end{vmatrix}$, using cofactor expansion:

Cofactor expansion along column 1:

$$\begin{vmatrix} 0 & 1 & 5 & 2 \\ 0 & 2 & 0 & 4 \\ 3 & 3 & 7 & 0 \\ 0 & 4 & 3 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 5 & 2 \\ 2 & 0 & 4 \\ 4 & 3 & 1 \end{vmatrix}$$

Continue with cofactor expansion along column 2:

$$3 \begin{vmatrix} 1 & 5 & 2 \\ 2 & 0 & 4 \\ 4 & 3 & 1 \end{vmatrix} = 3(-5) \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 3(-5)(2-16) = -15(-14) = 210$$

show why it is dismissed = 0

hmm... $\Rightarrow$ I understand your notation now

$$3 \begin{vmatrix} 1 & 5 & 2 \\ 2 & 0 & 4 \\ 4 & 3 & 1 \end{vmatrix} = 3(-3) \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 3(-3)(4-4) = -9(0) = 0$$

The determinant is $210 + 0 = 210$.

4. Use Gaussian elimination to show that the deter. of $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ is $(y-x)(z-x)(z-y)$:

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad \text{Subtract R1 from R2 and put in R2 and subtract R1 from R3 and put in R3.}$$

$$\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \quad \text{Divide R2 by } (y-x) \text{ and put in R2.}$$

$$(y-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & z-x & z^2-x^2 \end{vmatrix} \quad \text{Subtract } (z-x) \text{ R2 from R3 and put in R3.}$$

$$(y-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & (z^2-x^2)-(y+x)(z-x) \end{vmatrix} \quad \text{Div. R3 by } (z^2-x^2)-(y+x)(z-x) \text{ \& put in R3.}$$

$\leftarrow$ where does this come from

$$(y-x)(z-x)((z+x)-(y+x)) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & 1 \end{vmatrix}$$

The determinant is $1(y-x)(z-x)(z+y) = (y-x)(z-x)(z+y)$.

7

Let A be a 7×7 matrix and B the matrix that results when the columns of A are written in reverse order (last column becomes the first, and so forth). How are $\det(A)$ and $\det(B)$ related?

Good

Answer: It is known that If A is an $n \times n$ matrix and B is the matrix that results when two columns of A are interchanged, then $\det(B) = -\det(A)$. Therefore, if the matrix size is 7×7 , then it takes 3 column swaps to get B from A : Column 1 swap with 7, 2 with 6, and 3 with 5. Since the number of swaps is odd, $-1^3 = -1$. The $\det(B)$ will be the opposite of $\det(A)$, or $-\det(A)$.

non sequitur

1. Solve the system of equations by inverting the coefficient matrix and using the theorem that $A\mathbf{x} = \mathbf{b}$ has exactly one solution, that is $\mathbf{x} = A^{-1}\mathbf{b}$.

$$-b - 2c - 3d = 0$$

$$a + b + 4c + 4d = 7$$

$$a + 3b + 7c + 9d = 4$$

$$a - a - 2b - 4c - 6d = 6$$

Solution

Put the coefficients into a matrix augmented with the identity matrix.

$$\left[\begin{array}{cccc|cccc} 0 & -1 & -2 & -3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 1 & 3 & 7 & 9 & 0 & 0 & 1 & 0 \\ -1 & -2 & -4 & -6 & 0 & 0 & 0 & 1 \end{array} \right]$$

Interchange the R1 and R2, then multiply the new R2 by -1.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 3 & -1 & 0 & 0 & 0 \\ 1 & 3 & 7 & 9 & 0 & 0 & 1 & 0 \\ -1 & -2 & -4 & -6 & 0 & 0 & 0 & 1 \end{array} \right]$$

-(R1) $\rightarrow$ R3, then -(R1) $\rightarrow$ R4.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 5 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 & 0 & -1 \end{array} \right]$$

-2(R2) $\rightarrow$ R3, then -(R2) $\rightarrow$ R4.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & -2 & -1 & 1 & -1 & 0 & -1 \end{array} \right]$$

$2(R3) \rightarrow R4$.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array} \right]$$

$-(R4) \rightarrow R3$, $-(R4) \rightarrow R2$, and $-4(R4) \rightarrow R1$.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 0 & 12 & -3 & 8 & 4 \\ 0 & 1 & 2 & 0 & 8 & -3 & 6 & 3 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array} \right]$$

$-2(R3) \rightarrow R2$, $-4(R3) \rightarrow R2$.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 8 & -3 & 4 & 0 \\ 0 & 1 & 0 & 0 & 6 & -3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array} \right]$$

$-(R2) \rightarrow R1$. Now we have an augmented matrix with I on the left and A^{-1} .

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 6 & -3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array} \right]$$

With the given theorem $A^{-1}\mathbf{b} = \mathbf{x}$, we can set up the following equation:

$$\begin{bmatrix} 2 & 0 & 0 & -1 \\ 6 & -3 & 4 & 1 \\ 1 & 0 & 1 & 1 \\ -3 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ 4 \\ 6 \end{bmatrix} = \mathbf{x}, \text{ solving these we get } \mathbf{x} = \begin{bmatrix} -6 \\ 1 \\ 10 \\ -7 \end{bmatrix}$$

multiplying

So the final solution is: $a = -6$, $b = 1$, $c = 10$, and $d = -7$.

Prove that if A is invertible, $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

Answer:

We need to prove that if $A\mathbf{x} = \mathbf{b}$ has a solution, then $\mathbf{x} = A^{-1}\mathbf{b}$ is the solution.

Given $A\mathbf{x} = \mathbf{b}$, and that A^{-1} exists,

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}.$$

Using the associative property of matrix multiplication, this is equal to

$$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}.$$

Notice that $A^{-1}A = I$, so

$$I\mathbf{x} = A^{-1}\mathbf{b},$$

and any matrix multiplied by the identity matrix, I , results in that matrix, so

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Therefore, if A is invertible, $A\mathbf{x} = \mathbf{b}$ has exactly one solution, and that solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

doesn't follow just yet —
you showed that if there is
a solution, there is only one,
namely $\underline{x_c} = A^{-1}\mathbf{b}$. Thus $\underline{x_c} = A^{-1}\mathbf{b}$
is only a candidate for a solution (but
it is the only candidate). Now you
must show that the candidate is
successful, to which:

$$A\underline{x_c} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b},$$

Note the first time you use $A^{-1}A = I$,

the second time $AA^{-1} = I$.

$$A = \begin{vmatrix} 2 & 4 & 7 & 1 \\ 8 & 8 & 21 & 13 \\ 17 & 2 & 4 & 11 \\ 3 & 2 & 7 & 6 \end{vmatrix}$$

Find the inverse of matrix A:

Good

Answer:

A is a singular matrix because after performing the below operations R2 and R4 are equal to each other. A matrix with two rows that are equal to each other is singular.

$$A = \begin{vmatrix} 2 & 4 & 7 & 1 \\ 8 & 8 & 21 & 13 \\ 17 & 2 & 4 & 11 \\ 3 & 2 & 7 & 6 \end{vmatrix} \xrightarrow{R2-R1 \rightarrow R2} A = \begin{vmatrix} 2 & 4 & 7 & 1 \\ 6 & 4 & 14 & 12 \\ 17 & 2 & 4 & 11 \\ 3 & 2 & 7 & 6 \end{vmatrix} \xrightarrow{R2/2 \rightarrow R2} \begin{vmatrix} 2 & 4 & 7 & 1 \\ 3 & 2 & 7 & 6 \\ 17 & 2 & 4 & 11 \\ 3 & 2 & 7 & 6 \end{vmatrix}$$

An equation written :

$$ax^2 + bx + c = y$$

passes through points $(x, y) : (1, 1); (-1, -1); (-2, 4)$

find the values of a , b , and c .

answer :

written as a system of linear equations :

$$a(1) + b(1) + c(1) = 1$$

$$a(1) + b(-1) + c(1) = 1$$

$$a(4) + b(-2) + c(1) = 4$$

as an augmented matrix :

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 4 & -2 & 1 & 4 & 4 \end{array} \right] \begin{array}{l} R_2 - R_1 \rightarrow R_2 \text{ and } R_3 - 4R_1 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & -6 & -3 & 0 & 0 \end{array} \right] \begin{array}{l} R_3 - 3R_2 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & -3 & 6 & 0 \end{array} \right] \begin{array}{l} -\frac{1}{2} R_2 \rightarrow R_2 \text{ and } -\frac{1}{3} R_3 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array} \right] \text{so : } a=2, b=1, c=-2.$$

$$= -2 \begin{vmatrix} 1 & 3 & \frac{9}{2} & -2 \\ 0 & -6 & \frac{-23}{2} & 13 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 6 \end{vmatrix} \begin{matrix} R_2 \cdot \frac{-1}{6} \Rightarrow R_2, \\ R_3 \cdot \frac{-1}{3} \Rightarrow R_3, \\ R_4 \cdot \frac{1}{6} \Rightarrow R_4 \end{matrix}$$

$$= -2(-6)(-3)(6) \begin{vmatrix} 1 & 3 & \frac{9}{2} & -2 \\ 0 & 1 & \frac{-23}{12} & -\frac{7}{6} \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= -2(-6)(-3)(6)(1) = -216$$

given Matrix A such that:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

find the inverse of A and show that $A^{-1} \times A = I$.

answer:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \text{ and } R_3 - R_1 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + R_3 \rightarrow R_1 \text{ and } R_3 + 2R_2 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 0 & 1 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right] \begin{array}{l} R_1 + 8R_3 \rightarrow R_1 \text{ and } -3R_3 + R_2 \rightarrow R_2 \text{ and } -R_3 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{array} \right] \text{ so, } A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & 1 \end{bmatrix}$$

$$A^{-1} \times A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \times \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & 1 \end{bmatrix} =$$

$$\left[\begin{array}{ccc|ccc} (-40 + 26 + 15) & (16 - 10 - 6) & (9 - 6 - 3) \\ (-80 + 65 + 15) & (32 - 25 - 6) & (18 - 15 - 3) \\ (-40 + 0 + 40) & (16 - 0 - 16) & (9 + 0 - 9) \end{array} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1

Andrew Zitzelberger
Robert Park

Consider the matrix $A = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 7 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

a) Find A^{-1} , A^T , and $\det(A)$.

- show.

$$A^{-1} = \begin{pmatrix} -2 & 1 & 1 \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{5}{3} & -1 & \frac{1}{3} \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 7 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \det(A) = -3.$$

b) Find $(A^{-1})^T$, $(A^T)^{-1}$, and $\det(A^T)$.

- show *- why?*

$$(A^{-1})^T = \begin{pmatrix} -2 & \frac{1}{3} & \frac{5}{3} \\ 1 & 0 & -1 \\ 1 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad (A^T)^{-1} = \begin{pmatrix} -2 & \frac{1}{3} & \frac{5}{3} \\ 1 & 0 & -1 \\ 1 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad \det(A^T) = -3.$$

c) Show that for all invertible matrices A , $(A^{-1})^T = (A^T)^{-1}$.

assures result - doesn't show anything

$$\begin{aligned} (A^{-1})^T &= (A^T)^{-1} \\ A^T(A^{-1})^T &= A^T(A^T)^{-1} \\ I &= (A(A^{-1}))^T \\ I &= I^T \\ I &= I \end{aligned}$$

AND

- ditto see notes for proper demonstration

$$\begin{aligned} (A^{-1})^T &= (A^T)^{-1} \\ ((A^{-1})^T)A^T &= ((A^T)^{-1})A^T \\ I &= (A^{-1}A)^T \\ I &= I^T \\ I &= I \end{aligned}$$

d) Show that for all square matrices A , $\det(A) = \det(A^T)$.

The determinant of A can be found by cofactor expansion. The cofactor expansion along a row of A is the same expansion of the corresponding column of A^T . Thus $\det(A) = \det(A^T)$.

*But how do we know we can do
cofactor expansion along a column?
Again, no demonstration.*

Find The Inverse of Matrix(A) Using Gauss-Jordan Elimination

Matrix (A)

$$\begin{array}{ccc|c} 3 & 1 & 8 & \\ 1 & 10 & 4 & \\ 2 & 5 & 9 & \end{array}$$

Matrix (A)

$$\begin{array}{ccc|c} 3 & 1 & 8 & \\ 1 & 10 & 4 & \\ 2 & 5 & 9 & \end{array}$$

Matrix (I)

$$\begin{array}{ccc|ccc} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}$$

Switch R1 and R2

$$\begin{array}{ccc|ccc} 1 & 10 & 4 & 0 & 1 & 0 \\ 3 & 1 & 8 & 1 & 0 & 0 \\ 2 & 5 & 9 & 0 & 0 & 1 \end{array}$$

Add -3(R1) to R2

Add -2(R2) to R3

$$\begin{array}{ccc|ccc} 1 & 10 & 4 & 0 & 1 & 0 \\ 0 & -29 & -4 & 1 & -3 & 0 \\ 0 & -15 & -1 & 0 & -2 & 1 \end{array}$$

Add -2(R3) to R2

$$\begin{array}{ccc|ccc} 1 & 10 & 4 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 1 & -2 \\ 0 & -15 & -1 & 0 & -2 & 1 \end{array}$$

Add 15(R2) to R3

$$\begin{array}{ccc|ccc} 1 & 10 & 4 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 1 & -2 \\ 0 & 0 & -31 & 15 & 13 & -29 \end{array}$$

Add -10 (R2) to R1

$$\begin{array}{ccc|ccc} 1 & 0 & 24 & -10 & -9 & 20 \\ 0 & 1 & -2 & 1 & 1 & -2 \\ 0 & 0 & -31 & 15 & 13 & -29 \end{array}$$

Divide (R3) by -31

non standard ordering - interesting

$$\begin{array}{cccccc|c} 1 & 0 & 24 & -10 & -9 & 20 & \\ 0 & 1 & -2 & 1 & 1 & -2 & \\ 0 & 0 & 1 & -15/31 & -13/31 & 29/31 & \end{array}$$

possible to avoid
fractions longer

Add 2(R3) to R2

Add -24(R3) to R1

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 50/31 & -33/31 & -76/31 & \\ 0 & 1 & 0 & 1/31 & 5/31 & -33/31 & \\ 0 & 0 & 1 & -15/31 & -13/31 & 29/31 & \end{array}$$

Therefore The Inverse of Matrix A is:

$$\begin{array}{ccc|c} 50/31 & -33/31 & -76/31 & \\ 1/31 & 5/31 & -33/31 & \\ -15/31 & -13/31 & 29/31 & \end{array}$$

Prove that if A has an inverse, it is unique.

Pf: Assume both matrices B and C are inverses of A. So assume $AB=BA=I=AC=CA$.

Therefore since $BAC = \begin{cases} B(AC)=BI=B \\ (BA)C=IC=C \end{cases}$

no such assumption made
or needed

$B=C$, and since B and C were assumed to be different solutions for the inverse of A but were proven equal, the inverse of A, if it exists, is unique.

assumption of using 2 distinct
symbols is simply that they
may be different,
maybe not.

Prove the following theorem:

If A is invertible, then $\det(A^{-1}) = 1/\det(A)$.

Proof: We know that $AA^{-1} = I$, so $\det(AA^{-1}) = \det(I) = 1$. However, we also know that $\det(AA^{-1}) = \det(A^{-1})\det(A)$, so, it follows from these two expressions that $\det(A^{-1})\det(A) = 1$. Since A is invertible, $\det(A) \neq 0$ and all that we have to do to solve for $\det(A^{-1})$ is to divide both sides of the equation by $\det(A)$. This gives us the following equation: $\det(A^{-1}) = 1/\det(A)$, which is what we were asked to prove.

and even more general as well

8

1. Use Gauss - Jordan Elimination to solve this system of equations: (30 points):

$$\begin{aligned} 3x - y + 2z &= 6 \\ 4y - 2z &= 1 \\ x + 3y &= -2 \\ 2x + y + z &= 3. \end{aligned}$$

proper format: problem
solution
problem
solution
!

2. Find the determinants of the following (40 points, 20 points each):

a.

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 4 & 6 & 8 \\ 1 & 2 & 5 & 6 \\ 1 & 2 & 3 & 9 \end{bmatrix}$$

b.

$$B = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 4 & 7 & 9 \end{bmatrix}$$

3. 35 points total

a: 25 points

A = Find A^{-1} if A^{-1} exists.

b. 10 Points

A is _____.

- a) Symmetric
- b) Consistent
- c) Diagonal
- d) Upper triangular
- e) Lower triangular
- f) None of the above
- g) All of the above

including "none of the above"?

4. Given the following systems of equations (45 pts.)

System A

$$\begin{aligned} x_1 + 4x_2 + 6x_3 &= 1 \\ 3x_1 + x_3 &= 2 \\ 2x_2 + 5x_3 &= 3 \end{aligned}$$

System B

$$\begin{aligned} 2x_1 + 2x_2 &= \lambda x_1 \\ 6x_1 + 3x_2 &= \lambda x_2 \end{aligned}$$

- a) Find all signed elementary products of the coefficient matrix (A) (12pts.).
- b) Find the determinant of A (8 pts.).
- c) Is A invertible? (5 pts.).
- d) Express B in the form $(\lambda I - B)x = 0$ (12pts.).
- e) Find the eigenvalues of B (8 pts.).

is B a system
or a matrix?
we find eigenvalues
of matrices.

Solutions:

1.

Convert the equations to the augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 6 \\ 0 & 4 & -2 & 1 \\ 1 & 3 & 0 & -2 \\ 2 & 1 & 1 & 3 \end{array} \right]$$

Perform a series of elementary row operations to convert this to reduced - row echelon form. First switch the first and third rows and subtract three times the new first row from the third and two times the new first row from the second.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 4 & -2 & 1 \\ 0 & -10 & 2 & 0 \\ 0 & -5 & 1 & 7 \end{array} \right]$$

Then, divide the third row by -2 and switch the second and third rows.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 5 & -1 & 0 \\ 0 & 4 & -2 & 1 \\ 0 & -5 & 1 & 7 \end{array} \right]$$

Subtract the third row from the second row and then subtract four times the new second row from the third row and add five times the new second row to the fourth row.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 6 & 2 \end{array} \right]$$

Divide the fourth row by two and switch the third and fourth rows.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 5 \end{array} \right]$$

Subtract the fourth row from the third row and then subtract two times the new third row from the fourth row.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 13 \end{array} \right]$$

Without going any farther, we know that this matrix corresponds to the system:

$$x + 3y = -2$$

$$y + z = -1$$

$$z = -4$$

$$0 = 13.$$

Because the last equation is not correct, the system has no solution and is therefore inconsistent.

Awesome development - just like a text book.

2.

a. Perform the EROs to get it to the following

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{array}{l} R1-R2 \rightarrow R2 \\ R1-R3 \rightarrow R3 \\ R1-R3 \rightarrow R4 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

thus,

$$\text{Det}(A) = 2 \cdot 2 \cdot 3$$

$$= 12$$

b.

using row 1 to find cofactors

$$\text{Det}(B) = 2 \begin{vmatrix} 1 & 2 & -3 & 0 & 2 \\ 7 & 9 & 4 & 9 \end{vmatrix}$$

$$= -10 - 24$$

$$= -34$$

3.

Interchange Row 1 and Row 2

Multiply Row 1 by

Add -3 times Row 1 to Row 3

Multiply Row 2 by -1

Add -3 times Row 2 to Row 3

Multiply Row 3 by

Add - times Row 3 to Row 1

Add 1 times Row 2 to Row 1

$$A^{-1} =$$

Part B: B

4.

a) The coefficient matrix is:

$$\begin{bmatrix} 1 & 4 & 6 \\ 3 & 0 & 1 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 6 \\ 3 & 0 & 1 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 6 \\ 3 & 0 & 1 \\ 0 & 2 & 5 \end{bmatrix}$$

Its signed elementary products are as follows:

$$(1 \cdot 0 \cdot 5) = 0$$

$$-(1 \cdot 1 \cdot 2) = -2$$

$$\begin{aligned}(4 * 1 * 0) &= 0 & -(4 * 3 * 5) &= -60 \\ (6 * 3 * 2) &= 36 & -(6 * 0 * 0) &= 0\end{aligned}$$

b) The determinant is the sum of all signed elementary products (found in part (a)).
 $\det(A) = 0 + 0 + 36 - 2 - 60 - 0 = -26$

c) Yes. According to Theorem 2.3.6, if A is an $n \times n$ matrix and the $\det(A) \neq 0$, A is invertible.

d) The linear system

$$2x_1 + 2x_2 = \lambda x_1$$

$$6x_1 + 3x_2 = \lambda x_2$$

Can be written in matrix form as

$$\begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Where $B = \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

unmotivated - no B or x
to elaborate on.

Rewriting the system would give

$$\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 2 & -2 \\ -6 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Inserting the identity matrix, I , gives

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 2 & -2 \\ -6 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Simplifying gives the required form $(\lambda I - B)x = 0$.

$$\begin{bmatrix} \lambda - 2 & -2 \\ -6 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 2 & -2 \\ -6 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

e) As shown in part (d), $\lambda I - B$ equals

$$\begin{bmatrix} \lambda - 2 & -2 \\ -6 & \lambda - 3 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - 2 & -2 \\ -6 & \lambda - 3 \end{bmatrix}$$

Good

To find the eigenvalues solve $\det(\lambda I - B) = 0$ for λ .

$$\det(\lambda I - B) = \lambda^2 - 5\lambda - 6 = 0$$

Factoring gives

$$(\lambda - 6)(\lambda + 1) = 0$$

So the eigenvalues of B are $\lambda = 6$ and $\lambda = -1$.

Problem 1

Compute, by Gauss-Jordan elimination, a formula for the inverse of the 2-by-2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
What assumptions did you need to make during your computations?

Solution 1

Upon augmenting the matrix with a 2-by-2 Identity matrix we create the following augmented matrix

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} a & b & 1 & 0 \\ 0 & (ad-bc)/a & -c/a & 1/a \end{bmatrix} \text{ (if } a \neq 0 \text{)} \quad (-c/a) R1 + R2 \rightsquigarrow R2$$

$$\rightsquigarrow \begin{bmatrix} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{1}{ad-bc} \end{bmatrix} \text{ (if } a \neq 0, (ad-bc) \neq 0 \text{)} \quad \begin{array}{l} (1/a) R1 \rightsquigarrow R1 \\ (a/(ad-bc)) R2 \rightsquigarrow R2 \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & \frac{1}{a}(1 + \frac{bc}{ad-bc}) & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{1}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{1}{ad-bc} \end{bmatrix} \quad (-b/a) R2 + R1 \rightsquigarrow R2$$

so the inverse is $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ or $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

We assumed both pivots (a and $ad-bc$) were nonzero.

The case $a = 0$ (and $ad-bc = -bc \neq 0$) has to be done separately, because we need to swap rows:

$$\begin{bmatrix} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & d/c & 0 & 1/c \\ 0 & 1 & 1/b & 0 \end{bmatrix} \text{ (if } bc \neq 0 \text{)} \quad \text{ok, I see}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -d/bc & 1/c \\ 0 & 1 & 1/b & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{d}{0-bc} & \frac{-b}{0-bc} \\ 0 & 1 & \frac{-c}{0-bc} & \frac{0}{0-bc} \end{bmatrix}.$$

But we get the same formula for A^{-1} . *so, in general, what's*

The restriction?

A. $ad-bc \neq 0$

Find the values of x, y, and z using Gauss-Jordan Elimination

$$X + 5Y + 23Z = 141$$

$$3X + 16Y + 73Z = 447$$

$$2X + 12Y + 55Z = 335$$

Let's find the values of x, y, and z in this augmented matrix using Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 5 & 23 & 141 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 16 & 73 & 447 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 12 & 55 & 335 \end{bmatrix}$$

$$-2R_1 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 5 & 23 & 141 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 16 & 73 & 447 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 9 & 53 \end{bmatrix}$$

$$-3R_1 + R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 5 & 23 & 141 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 9 & 53 \end{bmatrix}$$

$$-2R_2 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 5 & 23 & 141 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 5 \end{bmatrix}$$

$$-4R_3 + R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 5 & 23 & 141 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 5 \end{bmatrix}$$

$$-23R_3 + R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 5 & 0 & 26 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 5 \end{bmatrix}$$

$$-5R_2 + R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 5 \end{bmatrix}$$

$$X = 6, Y = 4, Z = 5$$

Problem 3

When is the product of two symmetric matrices symmetric? Explain your answer.

Solution 3

AB is symmetric if and only if $AB = (AB)^T$, by definition. Yet,
 $AB = (AB)^T = B^T A^T = BA$ (because A and B are symmetric). In other words,
 AB is symmetric if and only if A and B commute.

Drive A^{-1} and $\det(A)$ from given information.

Show your work

$$\text{adj}(A) = \begin{bmatrix} (ei - fh), -(4h - 5h), (4f - 5e) \\ -(di - fg), (3i - g), -(3f - 5d) \\ (dh - eg), -(eh - 4g), (3e - 4d) \end{bmatrix}$$

$$A = \begin{bmatrix} 3, 4, 5 \\ *, *, * \\ *, *, * \end{bmatrix}$$

redundant?

- a) $\det(a)$, b) Find A^{-1} through using its adj and det.

Solution

- a) Since we have $\text{adj}(A)$, we can switch back $\text{adj}(A)$ to A^T

doesn't mean what you did.

$$\text{adj}(A) = \begin{bmatrix} (ei - fh), -(4h - 5h), (4f - 5e) \\ -(di - fg), (3i - g), -(3f - 5d) \\ (dh - eg), -(eh - 4g), (3e - 4d) \end{bmatrix}$$

$$\Rightarrow \text{adj}(A)^T = \begin{bmatrix} (ei - fh), -(4h - 5h), (4f - 5e) \\ -(di - fg), (3i - g), -(3f - 5d) \\ (dh - eg), -(eh - 4g), (3e - 4d) \end{bmatrix}$$

punctuation

We also know that the first row of the original matrix A. "3, 4, 5", so we can use these values to make $\det(A)$.

$$\det(A) = 3(ei - fh) + 4(di - fg) + 5(dh - eg)$$

? - where did this come from?

b) Use theorem $A^{-1} = \frac{1}{\det(A)} \times \text{adj}(A)$

$$A^{-1} = \frac{1}{[3(ei - fh) + 4(di - fg) + 5(dh - eg)]} \times \begin{bmatrix} (ei - fh), -(4h - 5h), (4f - 5e) \\ -(di - fg), (3i - g), -(3f - 5d) \\ (dh - eg), -(eh - 4g), (3e - 4d) \end{bmatrix}$$

(16)

1) Using matrices A,B,C,D,E compute the following (if possible):

Each part worth 6 points for a total of up to 36 points

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

bad notation for
delim. + matrix

a) $-3(D+2E)$

b) BA

c) $2A^T + C$

d) $B - B^T$

e) CC^T

f) $(C^TB)A^T$

Solution:

$$a) \quad 2E = \begin{bmatrix} 12 & 2 & 6 \\ -2 & 2 & 4 \\ 8 & 2 & 6 \end{bmatrix} \quad D + 2E = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 12 & 2 & 6 \\ -2 & 2 & 4 \\ 8 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix}$$

$$-3(D+2E) = \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$$

b) BA not defined. $(2 \times 2)(3 \times 2)$

$$c) \quad A^T = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad 2A^T = \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix}$$

$$2A^T + C = \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

$$d) \quad B^T = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} \quad B - B^T = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\text{e) } C^T = \begin{vmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{vmatrix} \quad CC^T = \begin{vmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{vmatrix} = \begin{vmatrix} \mathbf{21} & \mathbf{17} \\ \mathbf{17} & \mathbf{35} \end{vmatrix}$$

$$\text{f) } C^T = \begin{vmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{vmatrix} \quad C^TB = \begin{vmatrix} 4 & 5 \\ 16 & -2 \\ 8 & 8 \end{vmatrix} \quad A^T = \begin{vmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$(C^TB)A^T = \begin{vmatrix} \mathbf{12} & \mathbf{6} & \mathbf{9} \\ \mathbf{48} & \mathbf{-20} & \mathbf{14} \\ \mathbf{24} & \mathbf{8} & \mathbf{16} \end{vmatrix}$$

#2 - (40 points)

a) Find an augmented matrix for the following system of equations. (10 points)

$$2x_1 + x_2 + 3x_3 = 1$$

$$x_1 + x_3 = 1$$

$$2x_2 + x_3 = 1$$

$$x_2 + 2x_3 = 3$$

If this is written $Ax=b$, then
is not A ,

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \quad AI = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Alternatively, finding the inverse of your A is
inherent to solving
systems

b) Now, using elementary row operations, find the inverse of the augmented matrix. (20 points)

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & -2 & 0 & 0 \\ 0 & 0 & -3 & -5 & 0 & 0 & 1 & -2 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & -2 & 0 & 0 \\ 0 & 0 & -3 & -5 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 & -1 & -2 & 0 & 1 \end{array} \right] \rightarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 11 & -4 & 8 & 1 & 2 \\ 0 & 0 & 0 & 7 & -3 & 6 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 11 & -4 & 8 & 1 & 2 \\ 0 & 0 & 0 & 1 & -\frac{3}{7} & \frac{6}{7} & \frac{1}{7} & \frac{1}{7} \end{array} \right] \rightarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & \frac{3}{7} & \frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ 0 & 1 & 1 & 0 & \frac{4}{7} & -\frac{8}{7} & \frac{1}{7} & \frac{1}{7} \\ 0 & 0 & 1 & 0 & \frac{5}{7} & -\frac{10}{7} & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 0 & 1 & -\frac{3}{7} & \frac{6}{7} & \frac{1}{7} & \frac{1}{7} \end{array} \right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{2}{7} & \frac{11}{7} & \frac{3}{7} & -\frac{4}{7} \\ 0 & 1 & 0 & 0 & \frac{1}{7} & \frac{2}{7} & \frac{5}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & 0 & \frac{5}{7} & -\frac{10}{7} & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 0 & 1 & -\frac{3}{7} & \frac{6}{7} & \frac{1}{7} & \frac{1}{7} \end{array} \right]$$

$$A^{-1}I = \begin{bmatrix} \frac{-2}{7} & \frac{11}{7} & \frac{3}{7} & \frac{-4}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{5}{7} & \frac{-2}{7} \\ \frac{5}{7} & \frac{-10}{7} & \frac{-4}{7} & \frac{3}{7} \\ \frac{-3}{7} & \frac{6}{7} & \frac{1}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A(A^{-1}) = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{-2}{7} & \frac{11}{7} & \frac{3}{7} & \frac{-4}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{5}{7} & \frac{-2}{7} \\ \frac{5}{7} & \frac{-10}{7} & \frac{-4}{7} & \frac{3}{7} \\ \frac{-3}{7} & \frac{6}{7} & \frac{1}{7} & \frac{1}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

c) Referring to the elementary row operations above, represent the augmented matrix 'A', in Row-Echelon form. (10 points)

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -4 & 8 & 1 & 2 \\ \frac{-3}{7} & \frac{6}{7} & \frac{1}{7} & \frac{1}{7} \end{bmatrix}$$

3. Using the matrix

$$A = \begin{vmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{vmatrix}$$

- Find the $\det(A)$. Worth: 10 points
- Find the matrix of Cofactors. Worth: 10 points
- Find the $\text{adj}(A)$. Worth: 10 points
- Using the results found in a.-c., find A^{-1} . Worth: 10 points

$$\begin{aligned} \text{a. } \det(A) &= \begin{vmatrix} 5 & 2 & 2 \\ 3 & 8 & 9 \\ 3 & 2 & 2 \end{vmatrix} - 3 \begin{vmatrix} 2 & 2 & 2 \\ 1 & 8 & 9 \\ 1 & 2 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 9 \\ 1 & 3 & 2 \end{vmatrix} - 1 \begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} \\ &= 1[5 \begin{vmatrix} 8 & 9 \\ 2 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 9 \\ 3 & 2 \end{vmatrix} + 2 \begin{vmatrix} 3 & 8 \\ 3 & 2 \end{vmatrix}] - 3[2 \begin{vmatrix} 8 & 9 \\ 2 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 9 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 8 \\ 1 & 2 \end{vmatrix}] + \\ &\quad 1[2 \begin{vmatrix} 3 & 9 \\ 3 & 2 \end{vmatrix} - 5 \begin{vmatrix} 1 & 9 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}] - 1[2 \begin{vmatrix} 3 & 8 \\ 3 & 2 \end{vmatrix} - 5 \begin{vmatrix} 1 & 8 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}] \\ &= 1[5(16-18) - 2(6-27) + 2(6-24)] - 3[2(16-18) - 2(2-9) + 2(2-8)] + \\ &\quad 1[2(6-27) - 5(2-9) + 2(3-3)] - 1[2(6-24) - 5(2-8) + 2(3-3)] \\ &= 1[5(-2) - 2(-21) + 2(-18)] - 3[2(-2) - 2(-7) + 2(-6)] + 1[2(-21) - 5(-7) + 2(0)] - 1[2(-18) - 5(-6) + 2(0)] \\ &= 1[-10 + 42 - 36] - 3[-4 + 14 - 12] + 1[-42 + 35 + 0] - 1[-36 + 30 + 0] \\ &= 1[-4] - 3[-2] + 1[-7] - 1[-6] \\ &= -4 + 6 - 7 + 6 \\ &= 1 \end{aligned}$$

b. Cofactor matrix:

$$\begin{aligned} C_{11} &= 5(16-18) - 2(6-27) + 2(-18) = -4 \\ C_{12} &= 2(16-18) - 1(4-4) + 1(18-16) = -2 \\ C_{13} &= 2(6-27) - 1(10-6) + 1(45-6) = -7 \\ C_{14} &= 2(6-24) - 1(10-6) + 1(40-6) = -6 \\ C_{21} &= 3(16-18) - 3(2-2) + 3(9-8) = -3 \\ C_{22} &= 1(16-18) - 1(2-2) + 1(9-8) = -1 \\ C_{23} &= 1(6-27) - 1(6-3) + 1(27-3) = 0 \\ C_{24} &= 1(6-24) - 1(6-3) + 1(24-3) = 0 \\ C_{31} &= 3(4-4) - 1(10-6) + 1(10-6) = 0 \\ C_{32} &= 1(4-4) - 2(2-2) + 1(2-2) = 0 \\ C_{33} &= 1(10-6) - 2(6-3) + 1(6-5) = -1 \end{aligned}$$

$$C_{34} = 1(10-6) - 2(6-3) + 1(6-5) = -1$$

$$C_{41} = 3(18-16) - 1(45-6) + 1(40-6) = 1$$

$$C_{42} = 1(18-16) - 2(9-8) + 1(2-2) = 0$$

$$C_{43} = 1(45-6) - 2(27-3) + 1(6-5) = -8$$

$$C_{44} = 1(40-6) - 2(24-3) + 1(6-5) = -7$$

Matrix $A_c =$ ^{notation?}
$$\begin{vmatrix} -4 & 2 & -7 & 6 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 8 & -7 \end{vmatrix}$$

c. $\text{adj}(A) = A_c^T =$
$$\begin{vmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{vmatrix}$$

This can be found by simply transferring over the diagonal or by exchanging row 1 into column 1, row 2 into column 2, row three into column 3, and row 4 into column 4.

d. $A^{-1} = (1/\det(A))\text{adj}(A)$

Since we already found in part a. that $\det(A) = 1$, we can simplify this equation to find that

$$A^{-1} = (1/1)\text{adj}(A) = \text{adj}(A)$$

We already found the $\text{adj}(A)$ in part c., so we now that this is here equal to the inverse, or A^{-1} , so the answer is the same as c.

$$= \begin{vmatrix} -4 & 2 & -7 & 6 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 8 & -7 \end{vmatrix}$$

4. Equivalent Statements

- (a) A is invertible.
- (b) $Ax=0$ has only the trivial solution.
- (c) The reduced row echelon form of A is the Identity Matrix.
- (d) A is expressible as a product of elementary matrices.

Prove that these statements are equivalent.

Answer

(a) $\Rightarrow$ (b) :

Given A is invertible, show that $Ax=0$ has only trivial solutions.

If A is invertible, then there exists A^{-1} such that $AA^{-1}=I$. Now suppose that $Ax=0$. We will set x to be any solution of x for the equation.

$$\begin{aligned} Ax &= 0 \\ A^{-1}Ax &= A^{-1}0 \\ A^{-1}Ax &= 0 \\ Ix &= 0 \\ x &= 0 \end{aligned}$$

(b) $\Rightarrow$ (c) :

Given $Ax=0$ has only the trivial solutions, show that the reduced row echelon form of A is the identity matrix. If $Ax=0$, then $x=0, x=0 \dots x=0$. The matrix for these sets of equations are as follows.

$$\begin{bmatrix} A_{11}x_1 & A_{12}x_2 & \dots & A_{1n}x_n & 0 \\ A_{21}x_1 & A_{22}x_2 & \dots & A_{2n}x_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n1}x_1 & A_{n2}x_2 & \dots & A_{nn}x_n & 0 \end{bmatrix}$$

cal de sac?

The augmented matrix is

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} & 0 \\ A_{21} & A_{22} & \dots & A_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} & 0 \end{bmatrix}$$

In order for $x=0, x=0 \dots x=0$, the matrix must be reduced to the reduced row echelon form, and the column of zeros deleted.

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

logic

see book for proof

The identity matrix.

(c) $\Rightarrow$ (d) :

Given the reduced row echelon form of A is the Identity matrix, show that A is expressible as a product of elementary matrices.

PROOF: If the identity matrix is the reduced row echelon form of A, then A can undergo a set of elementary row operations to become the identity matrix. The Elementary row operations are really just elementary matrices, multiplied by A on the left.

So $E_k \dots E_2 E_1 A = I$

We know that elementary matrices are invertible, so we can multiply on the left by these inverses, giving

$$A = E_1^{-1} E_2^{-1} \dots E_k^{-1} I = E_1^{-1} E_2^{-1} \dots E_k^{-1}$$

Thus, A is a product of elementary matrices.

(d) $\Rightarrow$ (a)

Given A is expressible as a product of elementary matrices, and we know that elementary matrices are invertible, then we know that A is invertible, as it is a product of invertible matrices.

Good problem

13

A problem using the ideas of section 1.4

1. Let A be the matrix

$$\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

Find $p(A)$, given $p(x) = 2x^3 + x^2 - 3x + 4$.

Solution:

When A is a square matrix, and we are given a polynomial, $p(x) = a + bx + \dots + mx^n$, then $p(A)$ is defined as: $aI + bA + \dots + mA^n$. "I" is the identity matrix of A and is multiplied by any constant that does not have a variable attached. *consider notation*

So for this problem we start by substituting A in for x and replacing 4 with 4I. *what does that mean?*

$$p(A) = 2A^3 + A^2 - 3A + 4I$$

We are given matrix A and we know by inspection that matrix I is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We plug in these known matrices:

$$P(A) = 2 \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}^3 + \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}^2 - 3 \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exponents affect everything inside the matrix. So applying the exponents we have: *not true!!*

$$P(A) = 2 \begin{bmatrix} 27 & 8 \\ 64 & 3 \end{bmatrix} + \begin{bmatrix} 9 & 4 \\ 16 & 2 \end{bmatrix} - 3 \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplication outside the matrix also affects everything inside the matrix: *true!*

$$P(A) = \begin{bmatrix} 54 & 16 \\ 128 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 4 \\ 16 & 2 \end{bmatrix} - \begin{bmatrix} 9 & 6 \\ 12 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Numbers in the same row and column are added or subtracted: *true!*

$$P(A) = \begin{matrix} 54 + 9 - 9 + 4 & 16 + 4 - 6 + 0 \\ 128 + 16 - 12 + 0 & 6 + 2 - 3 + 4 \end{matrix} \rightarrow \begin{bmatrix} 58 & 14 \\ 132 & 9 \end{bmatrix}$$

probably not correct

2. Prove that if A^{-1} exists the linear system $Ax=b$ has one solution which is: $A^{-1}b$

Then solve the system:

$$2x_1 + 6x_2 + 6x_3 = 6$$

$$2x_1 + 7x_2 + 6x_3 = 3$$

$$2x_1 + 7x_2 + 7x_3 = 2$$

Solution:

Solve for A^{-1} by simultaneous method

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 - R_1 \longrightarrow R_2 \\ R_3 - R_1 \longrightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right] \begin{array}{l} 1/2 R_1 \longrightarrow R_1 \\ R_3 - R_2 \longrightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right] R_1 - 3R_2 - 3R_3 \longrightarrow R_1$$

$$\text{So } A^{-1}b = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ -3 \\ -1 \end{bmatrix}$$

where did you
do this?

Nowhere, as
far as I
can tell.

Problem 3: Find the Inverse of matrix A

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

•

• Finding the inverse of a matrix greater than 2x2

• You set it next to the identity matrix, and then solve for the identity matrix.

• When you do an operation on A you also do it on the identity.

doesn't work. what you think it needs

• Once you solve for the identity, what was originally the identity is now the inverse of the original matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step - one: } R1 - R2 \rightarrow R3 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$\text{Step - two: } R2 + R3 \rightarrow R3 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array} \right]$$

$$\text{Step - Three: } R3 / 2 \rightarrow R3 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array} \right]$$

$$\text{Step - four: } R2 - R1 \rightarrow R2 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array} \right]$$

$$\text{Step - five: } R1 - R3 \rightarrow R1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1/2 \end{array} \right]$$

$$A(\text{inverse}) = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

A problem using the ideas of section 2.4

4. Given the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

- Determine how many unique elementary products exist and list them, and
- Find $\det(A)$ using your elementary products and showing your work.

Solution:

- Since each elementary product can use only one value from each row and one value from each column, you could just make a list of possible permutations, starting with the first row, first column entry, which makes any other values from the first row or first column unusable for an elementary product, and then continue choosing entries that are possible, eliminating the remaining entries from used rows and columns, until you have a unique permutation made up of one entry from the first column, one entry from the second, and one entry from the third, never repeating the use of any column. You then repeat the process taking different routes until you find all unique elementary products which are:

$aei, afh, bdi, bfg, cdh, ceg \rightarrow$ total of 6 elementary products.

- Once you have all of your elementary products, you can use them to find the determinant, remembering that some of your elementary products will be even (positive signed) and others will be odd (minus signed—will be subtracted).

Using the following help, you can tell which are odd and which are even.

a	b	c	a	b
d	e	f	d	e
g	h	i	g	h

The forward diagonals aei , bfg , and cdh are your even elementary products, while the backward diagonals bdi , afh , and ceg are your odd elementary products. Even elementary products minus odd elementary products gives you the determinant

$$\det(A) = aei + bfg + cdh - bdi - afh - ceg.$$

- to form signed elementary products.