14

Problem 1

Let $\mathbf{V_1}=(X, -1, 4)$, $\mathbf{V_2}=(-3, 1, 3X)$, $\mathbf{V_3}=(6, -2, -4)$. Find two distinct sets of the form $S=\{\mathbf{V_1}, \mathbf{V_2}, \mathbf{V_3}\}$ such that S is NOT a basis for R^3. (Choose values for X)

Solution:
(30 Points)

For S to be a basis for R^3 it must both span R^3 and be linearly independent. So, we must choose X such that one of these criteria is not met. (If either criterion is not met, the other is also not met as we shall see.)

If S did span R^3, we know that an arbitrary vector $\mathbf{b}= (b_1, b_2, b_3)$ could be expressed as a linear combination $\mathbf{b}= c_1\mathbf{V_1} + c_2\mathbf{V_2} + c_3\mathbf{V_3}$. Expressing this equation in terms of components gives

$(b_1, b_2, b_3)= (X c_1 - 3c_2 + 6c_3, -c_1 + c_2 - 2c_3, 4c_1 + 3Xc_2 - 4c_3)$

Good!

or, equivalently

$$\begin{aligned} X c_1 - 3c_2 + 6c_3 &= b_1 \\ -c_1 + c_2 - 2c_3 &= b_2 \\ 4c_1 + 3Xc_2 - 4c_3 &= b_3 \end{aligned}$$

By Theorem 4.3.4 this system of equations is consistent for all b_1, b_2, b_3 if and only if the determinant of the corresponding coefficient matrix is non-zero. So we shall choose X such that the determinant is zero, thus making the system inconsistent and S not span R^3.

(in general

But first, if we wanted to show that S was linearly independent, we would show that the equation $c_1\mathbf{V_1} + c_2\mathbf{V_2} + c_3\mathbf{V_3} = 0$ has only the solution $c_1=c_2=c_3=0$. To do so, we form a system of equations as done previously

$$\begin{aligned} X c_1 - 3c_2 + 6c_3 &= 0 \\ -c_1 + c_2 - 2c_3 &= 0 \\ 4c_1 + 3Xc_2 - 4c_3 &= 0 \end{aligned}$$ (Note that the coefficient matrix here is the same as above)

and show that it only has the trivial solution. By picking X such that the determinant of the coefficient matrix of these equations is zero, we prove that these homogeneous equations have more than just the trivial solution (again, by Theorem 4.3.4) and, thus, S is linearly dependent.

$$\text{Det}\begin{bmatrix} X & -3 & 6 \\ -1 & 1 & -2 \\ 4 & 3X & -4 \end{bmatrix} = X(-4+6X) + 1(12-18X) + 4(6-6)=0$$

$$= 6X^2-22X+12 = 3X^2-11X+6 = (3X-2)(X-3)=0$$

Therefore $X = 3, 2/3$ and

$S_1= \{\mathbf{V_1}=(3, -1, 4), \mathbf{V_2}=(-3, 1, 9), \mathbf{V_3}=(6, -2, -4)\}$
$S_2= \{\mathbf{V_1}=(2/3, -1, 4), \mathbf{V_2}=(-3, 1, 2), \mathbf{V_3}=(6, -2, -4)\}$

Problem 2:

Determine whether $\mathbf{v}_1 = (1, 3, 7)$, $\mathbf{v}_2 = (-3, 2, 0)$, and $\mathbf{v}_3 = (2, 1, 3)$ span the vector space $\mathbb{R}^3$.
Then, in words, spatially describe the span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Solution:
(40 points total)

(35 points)

For that vector set to span $\mathbb{R}^3$, we must determine whether an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ can be expressed as a linear combination

$$\boldsymbol{b} = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + k_3\boldsymbol{v}_3$$

of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Expressing this equation in terms of components gives

$$\left(b_1, b_2, b_3\right) = k_1(1, 3, 7) + k_2(-3, 2, 0) + k_3(2, 1, 3)$$

or

$$k_1 - 3k_2 + 2k_3 = b_1$$
$$3k_1 + 2k_2 + k_3 = b_2$$
$$7k_1 + 3k_3 = b_3$$

The question now is reduced to determining whether the system is consistent for all values of b_1, b_2, and b_3. Knowing the equivalent statements, $A\mathbf{x}=\mathbf{b}$ is consistent for all $\mathbf{b}$ if $\det(A) \neq 0$.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & 2 & 1 \\ 7 & 0 & 3 \end{bmatrix}$$

$$det(A) = 1 \cdot 2 \cdot 3 + -3 \cdot 1 \cdot 7 + 2 \cdot 3 \cdot 0 - 2 \cdot 2 \cdot 7 - -3 \cdot 3 \cdot 3 - 1 \cdot 1 \cdot 0 = -16 \neq 0$$

Since the determinant of the coefficient matrix is nonzero, the linear system is consistent for all $\mathbf{b}$ and therefore the given vector set spans $\mathbb{R}^3$.

(5 points)

Since the linear system is consistent for all $\mathbf{b}$, this shows that some linear combination of the given vector set can produce any point in $\mathbb{R}^3$, meaning that the span of the vector set is every point in $\mathbb{R}^3$ and not a point, line, or plane.

Problem 3:

Use Wronskian to show that the set is linearly dependent for the set in $C^{(n-1)}(-\infty, \infty)$ *where* $n=3$:

$$e^{3x},\ -4x,\ -e^{3x}$$

Solution:
(30 points)

It's clear by Theorem 5.3.1(a) that it is linearly dependent, but the question asks to solve by Wronskian. So the first thing to do is construct a matrix. The first row will be without taking a derivative, then the second row with the first derivative, and the third row with the second derivative. Then solve for the determinant. Giving the following:

$$\begin{bmatrix} e^{3x} & -4x & -e^{3x} \\ 3e^{3x} & -4 & -3e^{3x} \\ 9e^{3x} & 0 & -9e^{3x} \end{bmatrix}$$

Then by solving for the determinant it will be clear if it's possible to be linearly independent.

$$det = 36e^{6x} + 108(x)e^{6x} + 0 - 36e^{6x} - 0 - 108(x)e^{6x} = 0$$

Since the determinant is zero this set is linearly dependent, because it doesn't meet the requirements to be independent. Namely it doesn't have only the trivial solution (or as was shown here the determinant is zero).

Problem (4):

Determine the standard matrix for the linear operator T: $\mathbf{R}^3 \to \mathbf{R}^3$ that first rotates a vector **v** counterclockwise about the y-axis through an angle of θ, then reflects the resulting vector about the xy-plane, and then projects that vector orthogonally onto the xz-plane. Hint: Remember to look at what happens to the standard basis vectors for $\mathbf{R}^3$, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, to find the standard matrix for each operator. Then multiply these standard matrices.

Solution:
(40 points)

The linear transformation T can be expressed as the composition

$$T = T_3 \circ T_2 \circ T_1$$

Where T_1 is the rotation about the y-axis, T_2 is the reflection about the xy-plane, and T_3 is the orthogonal projection on the xz-plane.

T_1, T_2, and T_3 are

good

$$[T_1] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \qquad [T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad [T_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We also know that

$$[T_3 \circ T_2 \circ T_1] = [T_3][T_2][T_1]$$

Thus, the standard matrix for T is $[T] = [T_3][T_2][T_1]$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 0 & 0 \\ -\sin\theta & 0 & -\cos\theta \end{bmatrix}$$

is this correct? I don't think so!

Question **:**

what is the Wronskian of $f_1 = x^2$, $f_2 = x^3$, $f_3 = x^4$?

does this form a linearly independent set?

Answer **:**

$$det\left(\begin{bmatrix} x^2 & x^3 & x^4 \\ 2x & 3x^2 & 4x^3 \\ 2 & 6x & 12x^2 \end{bmatrix}\right) = x^2\left(36x^4 - 24x^4\right) - x^3\left(24x^3 - 8x^3\right) + x^4\left(12x^2 - 6x^2\right)$$

$$= 12x^6 - 16x^6 + 6x^6$$

$$= 2x^6 \qquad \textit{yes, linearly independent.}$$

Find the standard matrix [T] for the composition of linear combinations on $\mathbf{R}^3$

> A rotation of 60° around the x-axis, followed by a projection onto the xz plane, followed by a dilation with a factor of k=2.

Solution

To find the matrix for each of the linear operations, first find the equation that corresponds to the linear operation.

The equations for a rotation of 60° around the x-axis are:

$$W_1 = x$$
$$W_2 = y\cos\theta - z\sin\theta$$
$$W_3 = y\sin\theta + z\cos\theta$$

Evaluated at $\theta = 60°$ we get

$$W_1 = x$$
$$W_2 = \frac{1}{2}y - \frac{\sqrt{3}}{2}z$$
$$W_3 = \frac{\sqrt{3}}{2}y + \frac{1}{2}z$$

From these equations we can create a matrix that describes the linear operation.

The standard matrix that results from these equations is:

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

The equations for a projection onto the xz plane are:

$$W_1 = x$$
$$W_2 = 0$$
$$W_1 = z$$

The standard matrix for these equations is:

$$[T_2]=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

The equations for a dilation with factor k=2 are:

$W_1 = kx$
$W_2 = ky$
$W_3 = kz$

The standard matrix for these equations is:

$$[T_3]=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

The linear transformation T can be expressed as the combinations T_1, T2, T3. This is equivalent to the matrix multiplication of the standard matrices in reverse order of the composition.

$T = T_1 \circ T_2 \circ T_3 = [T_3]\,[T_2]\,[T_1]$

$$= \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix} * \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{bmatrix} * \begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & \frac{-\sqrt{3}}{2}\\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & \sqrt{3} & 1\end{bmatrix}$$

The standard matrix for composition,

$$[T] = \begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & \sqrt{3} & 1\end{bmatrix}.$$

Of the following, which are subspaces of R^3 when a, b, and c are all real numbers? If not show why they are not subsets.

a) (a, 0, 0)

b) (1, a, b)

c) (sin a, cos b)

d) All vectors of the form (a, b, c) where b = 2a

e) All vectors of the form (a, b, c) where a + b = 1

Answers:

A and E are both subspaces:

For all of the answers d, e, f, and k are also real numbers, they are used to denote a second vector of the same form and scaler multiples

a) (a, 0, 0):

(a, 0, 0) + (b, 0, 0) = (a + b, 0, 0), which is in the subspace since a and b are both real numbers and the summation of two real numbers is also a real number. It is therefore closed under addition.

k * $\underline{v}$ = (ka, 0, 0) which is in the subspace since a, which is any real number, multiplied by another real number is still a real number. The vector is closed under scaler multiplication.

(a, 0, 0) is closed under both addition and scaler multiplication it is therefore a subspace of R^3.

b) (1, a, b)

(1, a, b) + (1, c, d) = (2, a + c, b + d) which is not in the matrix because the first entry is not a 1, as defined. Thus the matrix is not closed under addition.

K * $\underline{v}$ = (k, ka, kb) which is not in the subspace since the first entry is only "1" when k=1, and by definition to be a subspace it would need to be a "1" for all values of k. It is therefore not closed under scaler multiplication.

(1, a, b) is NOT a subspace of R^3 because it is not closed under addition or scaler multiplication.

c) (sin a, cos b) $\notin \mathbb{R}^3$, is the simplest explanation

(sin a, cos b) + (sin c, cos d) = (sin a + sin c, cos b + cos d) which is not in the subspace because for some values of a, b, c, and d, the answer can not be found using the original vector. {Example: if a = b = c = d = $\pi/4$, the summation of the two vectors would be (2 sin $\pi/4$, 2 cos $\pi/4$) which is not in the space since sin a $\neq$ 2 sin a, and in this case 2 sin $\pi/4$ cannot be found by any value of a for the entry sin a} It is therefore closed under addition.

K * $\underline{v}$ = (ksin a, k cos b) which is not in the subspace since both sine and cosine have a maximum value of positive or negative 1, and cannot therefore be multiplied by any real number and stay in that range. The subspace is closed under scaler multiplication.

(sin a, cos b) is not subspace of R^3 since it is not closed under either addition or scaler multiplication. & doesn't live in $\mathbb{R}^3$.

d) All vectors of the form (a, b, c) where b = 2a

(a, b, c) + (d, e, f) = (a + d, b + e, c + f) which is a vector of the same form. This can be shown since we know that (b + e) = 2 * (a + d) $\rightarrow$ (2a + 2d) = 2 * (a + d) $\rightarrow$ 2a + 2d = 2a = 2d. The vector is closed under addition.

K * $\underline{v}$ = (ka, kb, kc) which is also a vector of the same form since kb = 2ka, and by factoring out the "k's" we get the original equation b = 2a. The vector is closed under scaler multiplication.

The vector (a, b, c) where b = 2a is a subspace of R^3 since it is closed under addition and scaler multiplication.

e) (a, b, c) where a + b = 1

(a, b, c) + (d, e, f) = (a + d, b + e, c + f) which is not a vector of the same form since { (a + d) + (b + e) $\neq$ 1 since by using the communative property of addition we know that (a + b) + (d + e) = 1 + 1 = 2, which means that the vector is not closed under addition.

K * $\underline{v}$ = (ka, kb, kc) which is also not a vector of the same form since k * (a + b) $\neq$ 1, since (a + b) = 1 and k is any real number we know that the equation does not work for every value of k.

(a, b, c) were a + b = 1 is not a subspace of R^3 since it is not closed under either addition or scaler multiplication.

12

Complete the following:

(a) Find the standard matrix for the stated composition of linear operators on R^3.

An orthogonal projection on the xy-plane, followed by a reflection about the yz-plane, followed by a dilation with factor $k = 4$.

(b) Perform the transformations on the matrix, $\mathbf{A}=\begin{bmatrix}2\\1\\-7\end{bmatrix}$

(c) In words, if possible state what operations to perform find the inverse of the transformations done in part (a).

Answers:

Part (a):

To find the standard matrix, we must know that $T = T_n \circ T_{n-1} \circ \ldots \circ T_3 \circ T_2 \circ T_1$, and for the given three transformations, what T_1, T_2, and T_3 are represented by in matrix form.

$T_1 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ $T_2 = \begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$T_3 = \begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$

Therefore,

$T = (T_3)(T_2)(T_1) = \begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$

$T = \begin{bmatrix}-4&0&0\\0&4&0\\0&0&4\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} = \begin{bmatrix}-4&0&0\\0&4&0\\0&0&0\end{bmatrix}$

Part (b):

column column

To perform the transformations on the matrix, it is necessary is to multiply the matrix by the standard matrix of the transformations.

$[T][A] = \begin{bmatrix}-4&0&0\\0&4&0\\0&0&0\end{bmatrix}\begin{bmatrix}2\\1\\-7\end{bmatrix} = \begin{bmatrix}-8\\-4\\0\end{bmatrix}$

good - very geometric

Part (c):

It is not possible to do, because we do not know what was on the z-axis before we projected it onto the xy-plane, therefore, although we know that to undo the reflection on the yz-plane is another reflection on the yz-plane, and to undo the dilation of $k = 4$, is a contraction of $k = ¼$; we can not find the standard matrix to undo the transformations done in part (a) because of the orthogonal projection onto the xy-plane.

Prove:
If a set $S=\{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, then for every $\mathbf{v} \in V$, there exists a unique coordinate vector of V, $(c_1, c_2, \ldots, c_n) \in R^n$ such that $\mathbf{v}= c_1v_1+ c_2v_2+\ldots+c_nv_n$, in exactly one way.

Proof:
Since S spans V, it follows from the definition of a spanning set that every vector in V is expressible as a linear combination of the vectors in S. To see that there is only one way to express a vector as a linear combination of the vectors in S, suppose that some vector **v** can be written in the two ways:

$$\mathbf{v} = c_1v_1+ c_2v_2+\ldots+c_nv_n \text{ and } \mathbf{v} = k_1v_1+ k_2v_2+\ldots+k_nv_n$$

Subtracting the two equations gives us the following:

$$\mathbf{0}= (c_1- k_1)v_1+ (c_2- k_2)v_2+\ldots+(c_n- k_n)v_n$$

Since this is the linear combination of vectors in S, and are linearly independent, it is implied that:

$$c_1- k_1 = 0 \quad c_2- k_2 = 0 \quad \ldots \quad c_n- k_n = 0$$

From which we find that:

$$c_1 = k_1 \quad c_2 = k_2 \quad \ldots \quad c_n = k_n$$

Which tells us that the two expressions for **v** are the same, and thus is the expression is unique.

Problem:

The following linear systems have solutions spaces that are subspaces of R^3. Geometrically, each solution space must be the origin only, a line through the origin, a plane through the origin, or all of R^3. For each of the following linear systems, give the solutions and assign one of these geometric conditions.

a.) $3x + y + 3z = 0$
$2x + y + z = 0$
$-2x + 2y - 10z = 0$

b.) $3x - y - 3z = 0$
$9x - 3y - 9z = 0$
$6x - 2y - 6z = 0$

Solution:

a.)

Put the equations of the linear system into matrix form.

$$\begin{bmatrix} 3 & 1 & 3 \\ 2 & 1 & 1 \\ -2 & 2 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Create the augmented matrix.

$$\begin{bmatrix} 3 & 1 & 3 & 0 \\ 2 & 1 & 1 & 0 \\ -2 & 2 & -10 & 0 \end{bmatrix}$$

Row reduce:

R1 – R2 → R1

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ -2 & 2 & -10 & 0 \end{bmatrix}$$

2R1–R2→R2
2R1 + R3→R3

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix}$$

R2/-1

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix}$$

R3 – 2R2 → R3

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Set parameters and solve:

$z = t$ $\quad y - 3z = 0 \quad x + 2z = 0$

$y - 3t = 0 \quad x + 2t = 0$

$y = 3t \quad x = -2t$

The solutions are

$x = -2t \quad y = 3t \quad z = t$

These are equations for the line through the origin that are parallel to the vector $\mathbf{v} = (-2, 3, 1)$.

b.)

Put the equations of the linear system into matrix form.

$$\begin{bmatrix} 3 & -1 & -3 \\ 9 & -3 & -9 \\ 6 & -2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Create the augmented matrix.

$$\begin{bmatrix} 3 & -1 & -3 & 0 \\ 9 & -3 & -9 & 0 \\ 6 & -2 & -6 & 0 \end{bmatrix}$$

Row reduce:

R3 – 2R1 → R3

$$\begin{bmatrix} 3 & -1 & -3 & 0 \\ 9 & -3 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R2 – 3R1 → R2

$$\begin{bmatrix} 3 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R1/3

$$\begin{bmatrix} 1 & -\frac{1}{3} & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From way up here we have the first claim.

Set parameters and solve:

$$x - \frac{1}{3}y - z = 0$$

$z = t$ $\quad$ $y = s$ $\quad$ $x - \frac{1}{3}y - z = 0$

$$x - \frac{1}{3}s - t = 0$$

$$x = \frac{1}{3}s + t$$

The solutions are

$$x = \frac{1}{3}s + t \qquad y = s \qquad z = t$$

These are equations for the plane through the origin with $\mathbf{n} = (1, -\frac{1}{3}, -1)$ as a normal vector.

Question

For the following matrix, perform the following:

a) Find bases for the row and column spaces of matrix A, and then identify the rank of the matrix

b) Find a basis for the solution space of the linear system $\mathbf{Ax} = \mathbf{0}$ and then identify the nullity of the matrix.

$$A=\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Solution

a)

To find a basis for the row space of matrix A, we begin by reducing to row-echelon form.

$$A=\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \begin{matrix} R2 - 2*R1 \rightarrow R2 \\ R3 - 2*R1 \rightarrow R3 \\ R4 + R1 \rightarrow R4 \end{matrix}$$

$$\neq \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R3 - R2 \rightarrow R3$$

$$\neq \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 5.5.6, the row vectors with the leading 1's (the nonzero row vectors) from a basis for the row space of R. Therefore the basis vectors are:

$$R1 = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \end{bmatrix}$$

$R2 = [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6]$
$R3 = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5]$

– beautiful!

Theorem 5.5.6 also states that the column vectors with the leading 1's of the row vectors form a basis for the column space of R. It can be observed from the row-echelon form of matrix A that this includes columns 1, 3, and 5. It should be noted, however, that these column vectors must be read from the original matrix A (before row-reduction). We know that the columns will be the same as those with the leading 1's in the reduced echelon form because of Theorem 5.5.5b. There the basis vectors are:

$$C1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \quad C2 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} \quad C3 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

By definition, the common dimension of the row space and column space of a matrix is the rank of A. Therefore:

Rank(A) = 3

b)

We begin with the row-echelon form of matrix A found in part (a), and further reduce to reduced row-echelon form.

$$A \sim \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R1 - 5*R3 \text{ -> } R1 \\ R2 + 2*R3 \text{ - > } R2 \end{matrix}$$

$$A \sim \begin{bmatrix} 1 & -3 & 4 & -2 & 0 & -21 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} R1 - 4*R2 \text{ -> } R1$$

$$A \sim \begin{bmatrix} 1 & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the equation $\mathbf{Ax} = \mathbf{0}$, we can derive the following system of equations:

$X_1 - 3X_2 - 14X_4 - 37X_6 = 0$
$X_3 + 3X_4 + 4X_6 = 0$
$X_5 + 5X_6 = 0$

Solving for the leading variables we get:

$X_1 = 3X_2 + 14X_4 + 37X_6$
$X_3 = -3X_4 - 4X_6$
$X_5 = -5X_6$

It follows that the general solution of the system is:

$X_1 = 3r + 14s + 37t$
$X_2 = r$
$X_3 = -3s - 4t$
$X_4 = s$
$X_5 = -5t$
$X_6 = t$

Fabulous!

This is equivalent to:

$$\begin{bmatrix} X1 \\ X2 \\ X3 \\ X4 \\ X5 \\ X6 \end{bmatrix} = r \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 14 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 37 \\ 0 \\ -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

The three vectors on the right hand side form a basis for the solution space, giving us:

$$V1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad V2 = \begin{bmatrix} 14 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } V3 = \begin{bmatrix} 37 \\ 0 \\ -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

By definition, the dimension of the nullspace of a matrix is called the nullity of that matrix. Therefore:

Nullity(A) = 3

for a vector space

11

A)What does it mean to be closed under addition and scalar multiplication?

B)Prove that the following is closed under addition and scalar multiplication.

what? what type of addition? what type of scalar multiplication?

[a b c d]
[0 e f g]
[0 0 h i]
[0 0 0 j]

Solution: what is W?

A) (a)Two vectors, u and v, (in W) added together form a vector in W.
(b)Any scalar k multiplied by a vector in W lies in W.

B)

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix} + \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c & 2d \\ 0 & 2e & 2f & 2g \\ 0 & 0 & 2h & 2i \\ 0 & 0 & 0 & 2j \end{bmatrix}$$

Here we see that the result is of the same form - good

and

$$k \times \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix} = \begin{bmatrix} ka & kb & kc & kd \\ 0 & ke & kf & kg \\ 0 & 0 & kh & ki \\ 0 & 0 & 0 & kj \end{bmatrix}$$

is also of the same form. So,
This proves that upper triangular matrices are a subspace of Mnn and have closure under addition and scalar multiplication.

great sentence - something like it should be included in question

A)What does it mean to be closed under addition and scalar multiplication?

B)Prove that the following is closed under addition and scalar multiplication.

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$$

Q: Let $P_2 = \{polynomials_ f(x) = a_2x^2 + a_1x + a_0, _ where_ a_0, a_1, a_2 \in R\}$, and define addition and scalar multiplication as follows:

(a) Addition. Let $f(x) = a_2x^2 + a_1x + a_0, _ and_ g(x) = b_2x^2 + b_1x + a_0$. Then define $(f+g)(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$

(b) Scalar multiplication. Let $r \in R, _ f(x) = a_2x^2 + a_1x + a_0$. Define ... ?

Show that P_2 is a vector space.

Solution:

Closure follows easily from the definitions since the right-hand sides of the equations in (a) and (b) are polynomials of degree ≤ 2. For commutatively

$$(f+g)(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$$
$$_____ = (b_2 + a_2)x^2 + (b_1 + a_1)x + (b_0 + a_0)$$
$$_____ = (g+f)(x)$$

Associatively follows similarly. The $\theta_ is_ 0x^2 + 0x + 0$, which is the function $f(x) \equiv 0$. The additive inverse –f is simply $-a_2x^2 - a_1x - a_0$. The other axioms are easily verified.

→ If this is the case, and you don't want someone to go through all 10, it might be better to construct a system in which only 2 or 3 axioms fail, asking the student to find them.

1) Prove the Pythagorean Theorem in R^n

If **u** and **v** are orthogonal vectors in R^n with the Euclidean inner Product then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

SOLUTION

In order to prove the first result, note that if $\mathbf{u} = \mathbf{0}$ it is clear. Otherwise, we can write $\mathbf{v} = (\mathbf{v} \cdot \mathbf{u})\mathbf{u}/\|\mathbf{u}\|^2 + \mathbf{z}$ with the two terms on the right hand side orthogonal to each other. The Pythagorean Theorem yields $\|\mathbf{v}\|^2 = \|(\mathbf{v} \cdot \mathbf{u})\mathbf{u}/\|\mathbf{u}\|^2\|^2 + \|\mathbf{z}\|^2 \geq \|(\mathbf{v} \cdot \mathbf{u})\mathbf{u}/\|\mathbf{u}\|^2\|^2 = |\mathbf{v} \cdot \mathbf{u}|^2/\|\mathbf{u}\|^2$. Hence $\|\mathbf{v}\|^2\|\mathbf{u}\|^2 \geq |\mathbf{v} \cdot \mathbf{u}|^2$ and the result follows.

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \, \|\mathbf{v}\| + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

The triangle inequality can be deduced from the Cauchy-Schwarz inequality, because

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \, \|\mathbf{v}\| + \|\mathbf{v}\|^2 = (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.$$

which one?

how do you know?

not equal in general

I thought you were trying to prove P.T. in $\mathbb{R}^2$, not use it to produce some other result.

which result? doesn't sound like P.T. sounds like C.S.

Non sequitur

10

1. *40 points.* Find the standard matrix for the linear transformation T defined by the following formula. Define the domain and the codomain of T.

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -x_2 - x_3, x_1 + 3x_2 - x_3, x_2)$$

Solution: We have

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -x_2 - x_3, x_1 + 3x_2 - x_3, x_2).$$

This transformation is defined by the following equations:

$$\begin{aligned} w_1 &= 3x_1 + x_3 \\ w_2 &= -x_2 - x_3 \\ w_3 &= x_1 + 3x_2 - x_3 \\ w_4 &= x_2 \end{aligned}$$

It follows that T can be expressed in matrix form as

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

so the standard matrix for T is

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since T assigns a unique point (w_1, w_2, w_3, w_4) in R^4 to each point (x_1, x_2, x_3) in R^3, we can say $T : R^3 \to R^4$. Therefore, the domain of T is R^3 and the codomain of T is R^4.

2. *40 points.* Determine whether the linear operator $T : R^3 \to R^3$ defined by the following equations is one-to-one. If so, find the standard matrix for the inverse operator and find $T^{-1}(w_1, w_2, w_3)$.

$$\begin{aligned} w_1 &= 3x_1 + 2x_2 + x_3 \\ w_2 &= 4x_1 + 2x_2 \\ w_3 &= x_1 + x_2 - 3x_3 \end{aligned}$$

Solution: The matrix form of these equations is

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 0 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

so the standard matrix for T is

$$[T] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 0 \\ 1 & 1 & -3 \end{bmatrix}.$$

To determine whether T is one-to-one, we must find the determinant of T. By equivalent statements, if $\det[T] \neq 0$, T is invertible and T is one-to-one. We used cofactor expansion to find $\det[T]$:

$$\begin{aligned} \begin{vmatrix} 3 & 2 & 1 \\ 4 & 2 & 0 \\ 1 & 1 & -3 \end{vmatrix} &= 3\begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} - 2\begin{vmatrix} 4 & 0 \\ 1 & -3 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} \\ &= 3(-6) - 2(-12) + (4-2) \\ &= -18 + 24 + 2 = 8 \neq 0. \end{aligned}$$

Therefore, since $\det[T] \neq 0$, T is one-to-one.

There are several ways to determine T^{-1}. We chose to use the adjoint definition, where $T^{-1} = \frac{1}{det[T]} adj[T]$. We begin by finding the matrix of cofactors of T, which is

$$\begin{bmatrix} -6 & 12 & 2 \\ 7 & -10 & -1 \\ -2 & 4 & -2 \end{bmatrix}.$$

It follows that the adjoint of T is

$$\begin{bmatrix} -6 & 7 & -2 \\ 12 & -10 & 4 \\ 2 & -1 & -2 \end{bmatrix}$$

The standard matrix for T^{-1} is

$$[T^{-1}] = [T]^{-1} = \frac{1}{det[T]} adj[T] = \frac{1}{8}\begin{bmatrix} -6 & 7 & -2 \\ 12 & -10 & 4 \\ 2 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -3/4 & 7/8 & -1/4 \\ 3/2 & -5/4 & 1/2 \\ 1/4 & -1/8 & -1/4 \end{bmatrix}.$$

Thus

$$[T^{-1}]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -3/4 & 7/8 & -1/4 \\ 3/2 & -5/4 & 1/2 \\ 1/4 & -1/8 & -1/4 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -3/4w_1 & 7/8w_2 & -1/4w_3 \\ 3/2w_1 & -5/4w_2 & 1/2w_3 \\ 1/4w_1 & -1/8w_2 & -1/4w_3 \end{bmatrix}$$

from which we conclude that

$$T^{-1}(w_1, w_2, w_3) = (-3/4w_1 + 7/8w_2 - 1/4w_3, 3/2w_1 - 5/4w_2 + 1/2w_3, 1/4w_1 - 1/8w_2 - 1/4w_3).$$

3. *40 points.* Determine whether the set of all polynomials of the form $a_0 + a_1x + a_2x^2$, where $a_0, a_1, a_2 \in R$, is a subspace of P_2.

Solution: To begin, we let W equal the set of all polynomials of the form $a_0 + a_1x + a_2x^2$, where $a_0, a_1, a_2 \in R$. To determine whether W is a subspace of P_2, we must see if Axioms 1 and 6 both hold for any vectors in W. We first test Axiom 1:

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = a_0 + a_1x + a_2x^2 + b_0 + b_1x + b_2x^2$$

$$= a_0 + b_0 + a_1x + b_1x + a_2x^2 + b_2x^2 = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2.$$

Since $(a_0 + b_0), (a_1 + b_1), (a_2 + b_2) \in R$, $(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \in W$, so Axiom 1 holds. We now test Axiom 6:

$$k(a_0 + a_1x + a_2x^2) = ka_0 + ka_1x + ka_2x^2.$$

Since $ka_0, ka_1, ka_2 \in R$, $ka_0 + ka_1x + ka_2x^2 \in W$, so Axiom 6 holds. By Theorem 5.2.1, since both Axioms 1 and 6 hold, W is a subspace of P_2. — indeed $W = P_2$.

4. *30 points.* Use the Wronskian to show that the following set of vectors is linearly independent:

$$2, 5x, 3x^2$$

Solution: To show that the set of vectors is linearly independent using the Wronskian, we first define each vector as a function:

$$\mathbf{f}_1 = 2, \mathbf{f}_2 = 5x, \mathbf{f}_3 = 3x^2.$$

By definition, the Wronskian of $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ is the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix} = \begin{vmatrix} 2 & 5x & 3x^2 \\ 0 & 5 & 6x \\ 0 & 0 & 6 \end{vmatrix} = 2 \cdot 5 \cdot 6 = 30.$$

By Theorem 5.3.4, since the Wronskian does not equal zero on $(-\infty, \infty)$, these functions form a linearly independent set of vectors.

Math 343 Test, Chapters 4-5

(9)

1. (40 points) Prove the following theorem:

   Let $S = \{\mathbf{v_1}, \ldots, \mathbf{v_r}\} \subset \mathbb{R}^n$. If $r > n$, then S is linearly dependent.

   Solution:

   Let

$$\mathbf{v_1} = (v_{11}, v_{12}, \ldots, v_{1n}),$$
$$\mathbf{v_2} = (v_{21}, v_{22}, \ldots, v_{2n})$$
$$\vdots$$
$$\mathbf{v_r} = (v_{r1}, v_{r2}, \ldots, v_{rn})$$

   Consider the following equation:

$$k_1\mathbf{v_1} + k_2\mathbf{v_2} + \ldots + k_r\mathbf{v_r} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_r} \end{bmatrix}_{n\times r} \times \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{bmatrix}_{r\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$$

   Since $r > n$, the system will have ~~a~~ nontrivial solutions, so there will be at least one free variable, so the ~~system~~ is linearly dependent.

   ↑ ~~more~~ fewer equations (n)
   than unknowns (r).

2. (50 points) Prove that $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$

   Solution:

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}||^2 &= (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) \\ &= (\mathbf{u}\cdot\mathbf{u}) + 2\mathbf{uv} + (\mathbf{v}\cdot\mathbf{v}) \\ &\leq ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2|\mathbf{u}\cdot\mathbf{v}| \\ &\leq ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2||\mathbf{u}||\,||\mathbf{v}|| \\ &= (||\mathbf{u}|| + ||\mathbf{v}||)^2 \end{aligned}$$

   ← C.S. inequality

   Thus,

$$||\mathbf{u}+\mathbf{v}|| \leq (||\mathbf{u}+||\mathbf{v}||)^2$$

3. (50 points) Prove that all $n \times n$ matrices A such that $AB = BA$ for a fixed $n \times n$ matrix B are subspaces of M_{nn}.

   Solution:

$$A + A' = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} + \begin{bmatrix} a'_{11} & a'_{12} & \ldots & a'_{1n} \\ a'_{21} & a'_{22} & \ldots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n1} & a'_{n2} & \ldots & a'_{nn} \end{bmatrix}$$

Problem: Use Wronskian to show that the following sets of vectors are linearly independent.

a) $1, x, e^x$
b) $1, x, x^2$
c) cos(x), sin(x)

Solution:

a) $w[1, x, e^x] = \begin{bmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{bmatrix}$ $\det = e^x \neq 0$

Therefore, the set of vectors is linearly independent.

b) $w[1, x, x^2] = \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$ $\det = 1 \neq 0$

Therefore, the set of vectors is linearly independent.

c) w[cos(x), sin(x)] =

$$\begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix} \det = \cos^2(x) - (-\sin^2(x)) = \cos^2(x) + \sin^2(x) = 1 \neq 0$$

Therefore, the set of vectors is linearly independent.

Problem:

Let V be a vector space and **u** be a vector in V.

Prove:

$(-1)\mathbf{u} = -\mathbf{u}$

Solution:

If we want to show that $(-1)\mathbf{u} = -\mathbf{u}$, we must show that $\mathbf{u} + (-1)\mathbf{u} = 0$.

good

$$\begin{aligned} \mathbf{u} + (-1)\mathbf{u} &= 1\mathbf{u} + (-1)\mathbf{u} && \textbf{Axiom 10. } (1\mathbf{u} = \mathbf{u}) \\ &= (1 + (-1))\mathbf{u} && \textbf{Axiom 8. } ((k+m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}) \\ &= (0\mathbf{u}) \\ &= (0) \end{aligned}$$

← certain recent theorem

Problem:

Consider the vectors u= (5, -4, -6, 2) and v= (-4, -3, 1, 7). Find the Euclidean norm (distance) of u, the Euclidean norm of v, and the Euclidean distance between u and v. Are the vectors orthogonal? Are they dependent?

Solution:

llull=√((5^2)+(-4^2)+(-6^2)+(2^2)) = √(4+25+36+16) =√81 = **9**

llvll=√((-4^2)+(-3^2)+(1^2)+(7^2)) = √(16+9+1+49) = √75 = √3*25 = **5√3**

llu-vll = √((5- -4)^2+(-4- -3)^2+(-6-1)^2+(2-7)^2) = √((9^2)+(-1^2)+(-7^2)+(-5^2))
=√(81+1+49+25) =√156 =√4*39 = **2√39**

The vectors are orthogonal because the Euclidean inner product, or u·v, =0.

u·v = u1v1+u2v2+u3v3+u4v4 = (5*-4)+(-4*-3)+(-6*1)+(2*-5) = -20+12+-6+14 = 0

The vectors are not dependent since neither is a multiple of the other. They are independent.

A)What does it mean to be closed under addition and scalar multiplication?

B)Prove that the following is closed under addition and scalar multiplication.

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$$

Problem #1 40pts

Let T: R^3 $\longrightarrow$ R^3 be the reflection about the X Y plane. What are the eigenvalues for the standard matrix T.

SOLUTION: 10 pts for the standard matrix 15 pts for getting the determinant 15 pts for getting the correct eigenvalues.

Standard Matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \det(I\lambda - A) = \begin{pmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & \lambda+1 \end{pmatrix} = 0$$

Since this augmented matrix has only the trace, the determinant is the diagonal multiplied together.

Determinant is equal to $(\lambda-1)^2 * (\lambda+1) = \lambda^3-\lambda^2-\lambda+1 = 0$

The eigenvalues are $\lambda = 1, -1$

We can check to confirm these values by plugging them back into our matrix. When plugged back into the matrix we have the following.

If $\lambda = 1$

Then

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $\lambda = -1$

Then

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

After row Reduction you get the following values for your X Y Z

$X = s \quad Y = t \quad Z = 1$

$X = 1 \quad Y = 1 \quad Z = t$

Problem #2 40 pts

Determine if the vectors given form a linear independent or linear dependent set. If it is linear dependent, express each vector as a linear combination of the others.

A) V1 = <-2, 1, 3> V2 = <6, 0, 2> V3 = <4, 5,-1>

B) V1 = <-1, 2, 4> V2 = <1, 0, -2> V3 = <4, 2, 2>

SOLUTION: 20 pts for each part. 10 pts if they set up the augmented matrix correctly and 10 pts if they produce the correct results of linear dependency.

A) The equivalent equations are as follows:

$$-2k1 + 6k2 + 4k3 = 0$$
$$k1 + 5k3 = 0$$
$$3k1 + 2k2 - k3 = 0$$

The augmented matrix looks like:

$$\begin{pmatrix} -2 & 6 & 4 \\ 1 & 0 & 5 \\ 3 & 2 & -1 \end{pmatrix} * \begin{bmatrix} k1 \\ k2 \\ k3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Through a series of row reduction you will end up with the following matrix which only has the trivial solution as an answer therefore these three matrixes are linearly independent.

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{pmatrix} * \begin{bmatrix} k1 \\ k2 \\ k3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

B) The equivalent equations are as follows:

$$-k1 + 5k2 + 4k3 = 0$$
$$2k1 + 2k3 = 0$$
$$4k1 - 2k2 + 2k3 = 0$$

The augmented Matrix looks like:

$$\begin{pmatrix} -1 & 5 & 4 \\ 2 & 0 & 2 \\ 4 & -2 & 2 \end{pmatrix} * \begin{bmatrix} k1 \\ k2 \\ k3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Through a series of row reduction you will end up with the following matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} * \begin{bmatrix} k1 \\ k2 \\ k3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives us the following values for k1, k2, k3 respectively (-t, -t, t) where t is some arbitrary value. We can write our vector equations as $-tV1 - tV2 + tV3 = 0$. Since this is linearly dependent we can write each equation in terms of the others. These are:

$tV1 = -tV2 + tV3$
$tV2 = -tV1 + tV3$
$tV3 = tV1 + tV2$

Problem # 3 40 pts

Given the following equations and values, find the resultant matrix for the composition of linear operators on the vector (1, 2, 3)
The above mentioned vector goes through a rotation of 30 degrees about the x-axis, followed by a reflection about the XZ plane.

GIVEN:

X-axis rotation equations:

$W1 = X$
$W2 = Y\cos\theta - Z\sin\theta$
$W3 = Y\sin\theta + Z\cos\theta$

Reflection about the XZ plane:

$W1 = X$
$W2 = -Y$
$W3 = Z$

SOLUTION: 10 pts for creating the correct matrix for the rotation, 10 pts for the correct matrix for the reflection, 20 points for getting the correct matrix for the composition.

Augmented Matrix for 30 degree rotation:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & -3/2 \\ 0 & 1 & 3\sqrt{3}/2 \end{pmatrix}$$

Augmented Matrix for reflection:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

We were able to get these matrixes by using the equations given and plugging in our X, Y, Z values into the respective equations. Once we had the three equations we could put that in matrix form.

For the next part we need to know the definition of the composition of linear operators. Namely [Tc] = [T2]*[T1]. Since matrix multiplication is not communicative this is important that we take the second composition and multiply it by the first to get the total composition.

good - correct order

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & -3/2 \\ 0 & 1 & 3\sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2\sqrt{3} & 3 \\ 0 & 3 & 9\sqrt{3}/2 \end{pmatrix}$$

We just do plain matrix multiplication to get the complete composition.

Problem #4 30 pts

Using Theorem 5.2.1 determine if the following are subspaces

A) of R^3 all vectors of the form(a, 1, 1) — w the what addition & scalar multiplication?

B) of M2x2

$$\begin{pmatrix} A & A \\ -A & -A \end{pmatrix}$$

ditto

ditto

C) all polynomials of the form A0 + A1x where A0 and A1 are real numbers

SOLUTIONS 10 pts for each part
In order to solve these you need to know what Theorem 5.2.1 says. It defines a subspace based on axiom 1 and 6 which say if U and V are vectors in the vector space, then U+V has to be in the vector space. Also if k is some scalar and U is a vector in the vector space than kU has to also be in the vector space. If it fulfills those two conditions it is a subspace.

A) let U = (a`, 1, 1) and V = (a, 1, 1) Then U + V = (a`+a, 2, 2) This is not in the form (a, 1, 1) therefore it is not a subspace and we do not even need to test the other axiom.

B)

$$\begin{pmatrix} A & A \\ -A & -A \end{pmatrix} + \begin{pmatrix} A & A \\ -A & -A \end{pmatrix} = \begin{pmatrix} 2A & 2A \\ -2A & -2A \end{pmatrix}$$

This passes the first test because A can be any real number and if you multiply any real number by two it is still a real number.

$$K \begin{pmatrix} A & A \\ -A & -A \end{pmatrix} = \begin{pmatrix} KA & KA \\ -KA & -KA \end{pmatrix}$$

Once again any real number multiplied by a scalar is still a real number therefore this passes the second test and is a subspace.

C) Let U = (A0 + A1x) and V = (A0 + A1x) Then U + V = (2A0 + 2A1x) which is still in the set since A0 and A1 just have to be any real number. Lets test the second part k(A0 + A1x) = kA0 + kA1x and since k is just some real scalar variable then A0 and A1 will still be real numbers and therefore are in the set and this is a subspace.

5

1. Use the Wronskian to show that the functions form a linearly independent set of vectors in $C^1(-\infty, +\infty)$. (Each part worth 10 points for a total of 40 points)

   a) $f_1=1, f_2=x, f_3=e^x$
   b) $f_1=x, f_2=\sin x$
   c) $f_1=\sin x, f_2=\cos x, f_3=x\sin x$
   d) $f_1=e^x, f_2=xe^x, f_3=x^2e^x$

Solution:

$$W(x) = \begin{bmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{bmatrix}$$

using this,

a) $W(x) = \begin{bmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{bmatrix}$

The detA = $e^x \neq 0$ so this function does not have value zero for all x in the interval $(-\infty, +\infty)$ as can be seen by evaluating it at $x=1$ $(= e)$, so f_1, f_2, f_3 form a linearly independent set of vectors by Theorem 5.3.4.

b) $W(x) = \begin{bmatrix} x & \sin x \\ 1 & \cos x \end{bmatrix}$

The detA = $x\cos x - \sin x$. If evaluated at $x=\pi/2$, detA = $-1 \neq 0$ so this function does not have value zero for all x in the interval $(-\infty, +\infty)$. So f_1, f_2 form a linearly independent set of vectors by Theorem 5.3.4.

c) $W(x) = \begin{bmatrix} \sin x & \cos x & x\sin x \\ \cos x & -\sin x & \sin x + x\cos x \\ -\sin x & -\cos x & 2\cos x - x\sin x \end{bmatrix}$

The detA can be found by first simplifying using Gaussian Elimination:

$$W(x) \sim \begin{bmatrix} \sin x & \cos x & x\sin x \\ \cos x & -\sin x & \sin x + x\cos x \\ 0 & 0 & 2\cos x \end{bmatrix} \sim \begin{bmatrix} (\sin x)\begin{bmatrix} 1 & \cot x & x \\ \cos x & -\sin x & \sin x + x\cos x \\ 0 & 0 & 2\cos x \end{bmatrix} \end{bmatrix}$$

$$\sim (\sin x)\begin{bmatrix} 1 & \cot x & x \\ 0 & -\csc x & \sin x \\ 0 & 0 & 2\cos x \end{bmatrix} \sim (-\sin^{2x})\begin{bmatrix} 1 & \cot x & x \\ 0 & 1 & -\sin 2x \\ 0 & 0 & 2\cos x \end{bmatrix}$$

Thus the detA = $-2\cos x\sin^2 x \neq 0$ as can be seen by evaluating it at $x = \pi/4$ ($= -\sqrt{2}/2$) so this function does not have value zero for all x in the interval $(-\infty, +\infty)$, so f_1, f_2, f_3 form a linearly independent set of vectors by Theorem 5.3.4.

d) $W(x) = \begin{bmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x+xe^x & 2xe^x+e^x \\ e^x & 2e^x+xe^x & 3e^x+2xe^x \end{bmatrix}$

The determinant can be found by first simplifying using Gaussian Elimination:

$W(x) \sim \begin{bmatrix} e^x & xe^x & x^2e^x \\ 0 & e^x & 2xe^x+e^x-x^2e^x \\ 0 & 2e^x & 2xe^x+3e^x-x^2e^x \end{bmatrix} \sim$

$(e^x)\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x+1-x^2 \\ 0 & 2 & 2x+3-x^2 \end{bmatrix} \sim (e^x)\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x+1-x^2 \\ 0 & 0 & -2x+1+x^2 \end{bmatrix}$

Thus the detA = $(e^x)(-2x+1+x^2) \neq 0$ as can be seen by evaluating the determinant at x=0, detA=1, so this function does not have value zero for all x in the interval $(-\infty, +\infty)$, so f_1, f_2, f_3 form a linearly independent set of vectors by Theorem 5.3.4.

Good

2

40 points

Part (a) 10 points

Given the vector (2, 7), use matrix multiplication to find the image of the vector when reflected about the line $y = x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \Rightarrow (7, 2)$$

discussion?

Part (b) 10 points

Rotate the resulting vector from part (a) through an angle of 90^o.

$$\begin{bmatrix} \cos 90^o & -\sin 90^o \\ \sin 90^o & \cos 90^o \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix} \Rightarrow (-2, 7)$$

Part(c) 20 points

Now, project the resulting vector from part (b) orthogonally onto the $x-$axis.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \Rightarrow (-2, 0)$$

3. **Prove that the set of all pairs of real numbers of (x , y) with the operations (x , y) + (x* , y*) = (xx*, yy*) and k(x , y) = (kx, ky) form a vector space V, using the following ten axioms.**

1. **If u and v are objects in V, then u+v is in V.**
2. **u+v=v+u**
3. **u+(v+w)=(u+v)+w**
4. **There is an object 0 in V, called a zero vector for V, such that 0+u=u+0=u for all u in V.**
5. **For each u in V, there is an object -u in V, called a negative of u, such that u+(-u)=(-u)+u=0**
6. **If k is any scalar and u is any object in V, then ku is in V.**
7. **k(u+v)=ku+kv**
8. **(k+m)u=ku+mu**
9. **k(mu)=(km)(u)**
10. **1u=u**

Which axioms would you use to determine if something was a subset space **of this vector space V? What are the names of these axioms?**

good problem,
but wrong conclusion

4. Given the following set of polynomials:
f1 = 3 + x -4x^2
f2 = 2 + 5x + 6x^2
f3 = 1 + 4x + 8x^2

Determine if a) the set {f1, f2, f3} is linearly independent. (12 pts.)
b) the set {f1, f2, f3} spans P2. (12 pts.)
c) the set {f1, f2, f3} is a base for P2. (12 pts.)

a. We can find this out by making a matrix out of the system of equations such that
k1(3 + x -4x^2) + k2(2 + 5x + 6x^2) + k3(1 + 4x + 8x^2) = 0
Matrix A = and then we can row reduce…

$$\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} \begin{vmatrix} k1 \\ k2 \\ k3 \end{vmatrix} = 0$$

We know that if f1, f2, and f3 form a linearly dependent set if this system has a nontrivial solution and an independent set if the solution is the trivial solution. We then solve to find…

$$\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} \begin{matrix} -3R2 + R1 \rightarrow R1 \\ 4R2 + R3 \rightarrow R3 \\ R1 \leftrightarrow R3 \end{matrix}$$

$$\begin{vmatrix} 1 & 5 & 4 \\ 0 & -13 & -11 \\ 0 & 26 & 24 \end{vmatrix} 2R2 + R3 \rightarrow R3$$

$$\begin{vmatrix} 1 & 5 & 4 \\ 0 & -13 & -11 \\ 0 & 0 & 2 \end{vmatrix} 2R2 + R3 \rightarrow R3$$

2k3 = 0, so k3 = 0
-13 k2 + -11(0) = 0, so k2 = 0 also
k1 + 5(0) + 4(0) = 0, sp k1 = 0.

Thus, since k1=k2=k3=0, the solution is the trivial solution, and the set of equations {f1,f2,f3} must be linearly independent.

b) To see if this set {f1, f2, f3} spans P2, we must determine whether an arbitrary function b = b1 + b2x + b3x^2 in P2 can be expressed as a linear combination b = k1f1 + k2f2 + k3f3 of the functions f1, f2, and f3. Thus we get
b1 + b2x + b3x^2 = k1(3 + x -4x^2) + k2(2 + 5x + 6x^2) + k3(1 + 4x + 8x^2)
and we find in matrix form that

$$\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} \begin{vmatrix} k1 \\ k2 \\ k3 \end{vmatrix} = \begin{vmatrix} b1 \\ b2 \\ b3 \end{vmatrix}$$

where does it come from?

And we must determine if this system is consistant for all values of b1, b2, and b3. By Theorem 4.3.4 in the book, parts e. and g., the system is consistant for all b1, b2, and b3 iff the matrix has a nonzero determinate. To find the determinate, I will here use the shortcut of using different permutations and adding them since this is not the major subject of this exam. Thus…

$$\text{Det}\left(\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix}\right) \begin{matrix} 3 & 2 \\ 1 & 5 \\ -4 & 6 \end{matrix} = (3*5*8) + (2*4*-4) + (1*1*6) - (1*5*-4) - (3*4*6) - (2*1*8)$$

26 factorial?

$$= 120 - 32 + 6 + 20 - 72 - 16 = 26 \; != 0$$

Since the determinate of the matrix formed from the system of equations is nonzero, that means that the set of these functions spans P2.

c) The definition of a basis is that it is a set that is both linearly independent and spans the vector space that the set is a part of. Since we have already proved both of these qualities for this set {f1, f2, f3}, we know that it must be a basis of P2.

4

Show that the following polynomials are linearly dependant through finding the Wronskian and by showing that the constants in $a_1p_1+ a_2p_2+ a_3p_3 =0$ have more than just the trivial solution.

x^2+2x+2, $2x^2+x+1$, x^2+3x+3

Solution

$a_1 (x^2+2x+2)$
$a_2 (2x^2+x+1)$
$a_3 (x^2+3x+3)$ → $a_1x^2+ a_12x+ a_12+ a_22x^2+ a_2x+ a_2+ a_3x^2+ a_33x+ a_33=0$
Now we can put it into a matrix and then row reduce to show that the constants are not zero

[1 2 1] $R_2-2R_1>R_2$ [1 2 1] $R_2(-1/3)>R_2$ [1 2 1] $R_3-3R_2>R_3$ [1 2 1]
[2 1 3] $R_3-2R_1>R_3$ [0 -3 1] [0 1 -1/3] [0 1 -1/3]
[2 1 3] [0 -3 1] [0 -3 1] [0 0 0]

[1 2 1] $R_1-2R_2>R_1$ [1 0 5/3]
[0 1 -1/3] [0 1 -1/3]
[0 0 0] [0 0 0]

-5/3s

Therefore $a_1=0$, $a_2=1/3s$, $a_3=s$ which makes the equations linearly dependent. (20 pts)

Now we will show that the wronskian is equal to zero.

$$W(x)= \begin{bmatrix} x^2+2x+2 & 2x^2+x+1 & x^2+3x+3 \\ 2x+2 & 4x+1 & 2x+3 \\ 2 & 4 & 2 \end{bmatrix} =(-2x-2)(4x^2+2x+2-4x^2-12x-12)$$

$=(-2x-2)(4x^2+2x+2-4x^2-12x-12)+(4x+1)(2x^2+4x+4-2x^2-6x-6)-(2x+3)(4x^2+8x+8-4x^2-2x-2)$
$=(-2x-2)(-10x-10)+(4x+1)(-2x-2)+(-2x-3)(6x+6)$
$= 20x^2+40x+20-8x^2-10x-2-12x^2-30x-18= 20x^2-8x^2-12x^2+40x-10x-30x+20-2-18=0$
According to theorem 5.3.4 the equations are linearly dependent because the Wronskian is equal to zero. (20 pts)
Total pointt value is 40 Pts

This is the classic misreading of the theorem: a theorem does not imply its converse.

4.1 to 4.2

Worth 40 Pts.

Find the Matrix for the stated composition of linear operators on R^3.

A. A dilation with a factor of 4, followed by a rotation of 90^o about the x- axis.

B. An orthogonal projection on the xz-plane, followed by a rotation of 45^o about the z-axis, followed by a reflection about the xy-plane, followed by contraction with a factor of ½.

Answer Key:

A.

Step 1 – Multiply a std. RREF matrix by a factor of 4.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} * 4 = \begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$$

Std. RREF Matrix * Factor of 4

Step 2 – Multiply result from step 1 by the rotation about the x-axis operator with 90^o substituted in for Θ.

$$\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix} * \begin{bmatrix}\cos 90^0 & 0 & \sin 90^0\\0&1&0\\-\sin 90^0 & 0 & \cos 90^0\end{bmatrix} =$$

x-axis operator (substitutes inclusive)

$$\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix} * \begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix} = \begin{bmatrix}0&0&4\\0&4&0\\-4&0&0\end{bmatrix}$$

Values for above Solution

Answer:

$$\begin{bmatrix}0&0&4\\0&4&0\\-4&0&0\end{bmatrix}$$

B.

Step 1 – Muliply an orthogonal projection on the xz-plane operator by the rotation about the z-axis operator with 45^o substituted in for Θ.

wrong order - might make a difference.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos 45^0 & -\sin 45^0 & 0 \\ \sin 45^0 & \cos 45^0 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

xz-plane operator * rotation about z-axis operator

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

xz-plane operator * rotation about z-axis operator

wrong order

Step 2 – Multiply above result by the reflection about the xy-plane operator.

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

xy-plane operator

wrong order, but inconsequential here.

Step 3 – Multiply above result by a factor of ½.

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} * ½ = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Factor or ½

Answer:

$$\begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.a. Is the set of all the 2 × 2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$, with addition defined by

$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix} = \begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix}$ and scalar multiplication defined by

$k\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix}$ a vector space? Explain your reasoning.

b. If you answered yes in part a, what is the zero vector for this vector space? If you answered no, how could the form of the matrices could be changed to make this a vector space?

I would like to see 10 axioms, but think you may be correct.

Solution:
a. Yes, the set is a vector space, because all of the ten axioms that define a vector space hold.

Examples could be shown of each of the axioms, but in reality, this simple explanation should receive the full 20 points.

_____/20 (10 for the correct answer, 10 for the explanation)

b. The zero vector would be

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

If they answered no in part a and they have some logical explanation (probably involving changing the 1s in the matrix or the form of the addition and multiplication) they can receive up to 10 (pity) pts

_____/20

great problem, but would require 10 axioms to be demonstrated.

30 points possible

Let $\underline{v}_1 = (2, 4, 2)$ $\underline{v}_2 = (4, 18, 0)$ and $\underline{v}_3 = (6, 6, 8)$. Show that set $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is basis for R^3.

Solution

To show set S spans R^3, then need to show an arbitrary vector $\underline{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination

$$\underline{b} = c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3$$

$$(b_1, b_2, b_3) = c_1 (2, 4, 2) + c_2 (4, 18, 0) + c_3 (6, 6, 8)$$

$$(b_1, b_2, b_3) = (2c_1 + 4c_2 + 6c_3, 4c_1 + 18c_2 + 6c_3, 2c_1 + 0 + 8c_3)$$

Equating corresponding components,

$$\begin{aligned} 2c_1 + 4c_2 + 6c_3 &= b_1 \\ 4c_1 + 18c_2 + 6c_3 &= b_2 \\ 2c_1 \quad\quad + 8c_3 &= b_3 \end{aligned}$$

To show S spans R^3 then the system must have a solution for all $\underline{b}$.

To show linear independence must show only solution of

$$c_1\underline{v}_1 + c_2\underline{v}_2 + c3\underline{v}3 = \underline{0}$$

$$\text{Is } c_1 = c_2 = c_3 = 0$$

$$c_1 (2, 4, 2) + c_2 (4, 18, 0) + c_3 (6, 6, 8) = \underline{0}$$

$$(2c_1 + 4c_2 + 6c_3, 4c_1 + 18c_2 + 6c_3, 2c_1 + 0 + 8c_3) = (0, 0, 0)$$

Great insight.

Equating corresponding components,

$$\begin{aligned} 2c_1 + 4c_2 + 6c_3 &= 0 \\ 4c_1 + 18c_2 + 6c_3 &= 0 \\ 2c_1 \quad\quad + 8c_3 &= 0 \end{aligned}$$

Notice the coefficient matrix of both systems of equations is the same. Therefore can simultaneously show that first system is consistent for all $\underline{b}$ and the second system has only the trivial solution if the determinant of the coefficient matrix is not equal to zero.

$$\det\begin{vmatrix} 2 & 4 & 6 \\ 4 & 18 & 6 \\ 2 & 0 & 8 \end{vmatrix} = (2*18*8) + (4*6*2) + (6*4*0) - (6*18*2) - (4*4*8) - (2*6*0) = -8 \neq 0$$

So S is a basis for R^3.

(3)

Problem: Determine whether the following vectors

$$\mathbf{u} = (0,4,3) \quad \mathbf{v} = (1,-8,2) \quad \mathbf{w} = (0,12,4)$$

form a linearly dependent or a linearly independent set.

Solution: The vector equation "as a linear combination"

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$$

becomes

$$a(0,4,3) + b(1,-8,2) + c(0,12,4) = (0,0,0)$$

or, equivalently,

$$(b,\ 4a - 8b + 12c,\ 3a + 2b + 4c) = (0,0,0)$$

Setting corresponding components equal to each other gives

$$\begin{aligned} b &= 0 \\ 4a - 8b + 12c &= 0 \\ 3a + 2b + 4c &= 0 \end{aligned}$$

Substitution for b gives

$$\begin{aligned} 4a + 12c &= 0 \\ 3a + 4c &= 0 \end{aligned}$$

Thus, **u**, **v**, and **w** form a linearly dependent set if this system has a nontrivial solution, or a linearly independent set if it has only the trivial solution. Solving for the determinant gives

$$4(4) - 12(3) = -20$$

not equal to zero, so by our equivalent statements, this system has only the trivial solution, and is therefore a linearly independent set.

(30 points)

Problem:

If able, use the Wronskian to determine if the following functions form a linearly independent, or linearly dependent set of vectors in $C^2(-\infty, \infty)$. Explain your answer.

$x^2, 2x, 2$

Solution:

$$\text{Det}\begin{bmatrix} x^2, 2x, 2 \\ 2x, x, 0 \\ 2, 1, 0 \end{bmatrix} = 4x - 4x = 0$$

Not able to tell. $W(x^2, 2x, 2)=0$ so it does not tell us anything. If the Wronskian is not equal to zero we know the vectors are linearly independent but otherwise we are not sure either way. It is clear to see from inspection that these vectors are linear though.

$= -\begin{vmatrix} x^2 & 2 \\ 2x & 0 \end{vmatrix} = -(-1)4x = 4x \neq 0$ so independent,

The Linearly dependent part of this question is actually a trick question. We are not able to use the Wronskian to determine Linear dependance with what we know now.

(40 points)

Problem:

Given the equations: $w_1 = 4x + 3y + z$

$w_2 = 2x + y$

$w_3 = 3x + 2y + 2z$

determine if the linear operator $T: R^3 \rightarrow R^3$ defined by the equations is one-to-one. If so, find the standard matrix for the inverse operator.

Solution:

$$T = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 2 \end{bmatrix}$$

$\det T = 4(1)(2) + 2(2)(1) - 1(1)(3) - 3(2)(2) = 8 + 4 - 3 - 12 = -3$

Because $\det T \neq 0$, the linear operator T is one-to-one. To find the inverse operator, we find the inverse of T.

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 2 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_1 - R_2 \rightarrow R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & -1 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_3 - 4R_1 \rightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & -1 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -7 & 1 & 4 & -4 \end{array}\right] \quad R_1 + R_3 \rightarrow R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & 3 & -3 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -7 & 1 & 4 & -4 \end{array}\right] \quad R_2 - 2R_1 \rightarrow R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & 3 & -3 \\ 0 & 1 & 10 & -2 & -5 & 6 \\ 0 & -1 & -7 & 1 & 4 & -4 \end{array}\right] \quad R_3 + R_2 \rightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & 3 & -3 \\ 0 & 1 & 10 & -2 & -5 & 6 \\ 0 & 0 & 3 & -1 & -1 & 2 \end{array}\right] \quad \tfrac{1}{3} R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & 3 & -3 \\ 0 & 1 & 10 & -2 & -5 & 6 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \quad \begin{array}{l} R_1 + 5R_3 \to R_1 \\ R_2 - 10R_3 \to R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & 4/3 & \frac{1}{3} \\ 0 & 1 & 0 & -16/3 & -25/3 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

Thus the standard matrix of the inverse operator is:

$$T^{-1} = \begin{bmatrix} -\frac{2}{3} & 4/3 & \frac{1}{3} \\ -16/3 & -25/3 & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

(30 points)

Two-part Problem:

a.) Find the Euclidean inner product and determine whether the given vectors are orthogonal.

i.) $u= (0,3,-2,1)$ $v= (5,2,-1,0)$

ii.) $u= (9,4,-2,5)$ $v= (3,3,7,-5)$

iii.) $u=(a, b,-c, d)$ $v=(c, d, a, -b)$

solution:

i.) i.) $u= (0,3,-2,1)$ $v= (5,2,-1,0)$
$u\cdot v= (0\cdot 5) + (3\cdot 2) + (-2\cdot -1) + (1\cdot 0)$
$u\cdot v= 0 + 6 + 2 + 0$
$u\cdot v= 8$; not orthogonal

ii.) $u= (9,4,-2,5)$ $v= (3,3,7,-5)$
$u\cdot v= (9\cdot 3) + (4\cdot 3) + (-2\cdot 7) + (5\cdot -5)$
$u\cdot v= 27+12-14-25$
$u\cdot v=0$; orthogonal

iii.) $u=(a, b,-c, d)$ $v=(c, d, a, -b)$
$u\cdot v= ac + bd - ac - bd$
$u\cdot v= 0$; orthogonal

b.) Find the domain and codomain of the transformation defined by the equations, and determine whether the transformation is linear.

i.) $w_1 = 3x_1 - 2x_2 + 4x_3$
$w_2 = 5x_1 - 8x_2 + x_3$

ii.) $w_1 = x_1^2 - 3x_2 + x_3 - 2x_4$
$w_2 = x + 3x_1x_2 - 7x_3 + x_4$

solution:

i.) $T: R_3 \rightarrow R_2$, linear

ii.) $T: R_4 \rightarrow R_2$, nonlinear

(50 points)

2

Let V be a vector space, and **u** a vector in V.

Prove: $(-1)\mathbf{u} = -\mathbf{u}$

Answer:

In order to show that $(-1)\mathbf{u} = -\mathbf{u}$, we need to demonstrate that $\mathbf{u} + -\mathbf{u} = \mathbf{0}$.

this holds by definition of $-\underline{u}$ — no need to show that.

$\underline{u} + (-1)\underline{u} = \underline{0}$

This is what needs to be shown.

By Axiom 10, we know that
$\mathbf{u} + (-1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u}$
Then by Axiom 8, this is equal to
$(1 + (-1))\mathbf{u}$
Which, by scalar addition, is equal to
$0\mathbf{u}$
Finally, because of the multiplicative property of zero, this is equal to
$\mathbf{0}$.

Thus, $(-1)\mathbf{u} = -\mathbf{u}$.

This start indicates that you indeed wanted to show that $\underline{u} + (-1)\underline{u} = \underline{0}$, rather than the tautology $\underline{u} + -\underline{u} = \underline{0}$ — good.

Let $\mathbf{v_1} = (1, 3, 6)$, $\mathbf{v_2} = (2, 4, 2)$, $\mathbf{v_3} = (9, 6, 3)$. Determine whether those vectors span the vector space R^3.

We must show that a vector $\mathbf{b}= (b_1,b_2,b_3)$ can be expressed as a linear combination of
$\mathbf{b}=k_1\mathbf{v_1}+k_2\mathbf{v_2}+k_3\mathbf{v_3}$
$(b_1,b_2,b_3)= k_1(1, 3, 6)+ k_2(2, 4, 2)+ k_3(9, 6, 3)$
or
$1k_1+ 2k_2+ 9k_3= b_1$
$3k_1+ 4k_2+ 6k_3= b_2$
$6k_1+ 2k_2+ 3k_3= b_3$

$$A=\begin{bmatrix} 1 & 2 & 9 \\ 3 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}$$

$\det(A)= -108$

Therefore by equivalence theorem the system is consistent and the vectors span R^3.

QUESTION:
Let $\mathbf{c} = (-2, 3, 0, 6)$. Find all scalars b such that $\|b\mathbf{c}\| = 5$.

ANSWER:
$\|b\mathbf{c}\| = 5$
$|b| \cdot \|\mathbf{c}\| = 5$
$|b| \cdot \sqrt{(\mathbf{c} \cdot \mathbf{c})} = 5$
since $\mathbf{c} \cdot \mathbf{c} = (-2, 3, 0, 6) \cdot (-2, 3, 0, 6) = (4, 9, 0, 36)$, then
$|b| \cdot \sqrt{(4 + 9 + 0 + 36)} = 5$
$|b| \sqrt{(49)} = 5$
$7|k| = 5$
$|k| = 5/7$
$k = \pm 5/7$

Midterm Exam 2

Question 1

Part A (25 points):

List the standard matrices for the following operators on R^2.

(a) Rotation through an angle ϑ
(b) Dilation
(c) Orthogonal projection on the xy-plane
(d) Orthogonal projection on the xz-plane
(e) Orthogonal projection on the x-axis
(f) Orthogonal projection on the y-axis

Part B (20 points):

Find the standard matrix for the stated composition of linear operators on R^2.

(a) A rotation of 60 degrees, followed by a dilation with factor k = 4.
(b) A rotation of 60 degrees, followed by a rotation of 30 degrees, followed by a rotation of -45 degrees.

Solutions:

Part A:

(a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(b) $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Part B:

(a) $\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 2 \end{bmatrix}$

(b) $\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

Question 3 (40 Points)

Show that the functions $\mathbf{f_1 = x^2}$, $\mathbf{f_2 = e^x}$, and $\mathbf{f_3 = (x+2)}$ form a linearly independent set of vectors in C^2 on the interval from $(-\infty, +\infty)$.

Solution:

We will use the Wronskian to prove this set to be linearly independent.

$\mathbf{(f_1) = x^2}$	$\mathbf{(f_2) = e^x}$	$\mathbf{(f_3) = (x+2)}$
$\mathbf{(f_1)' = 2x}$	$\mathbf{(f_2)' = e^x}$	$\mathbf{(f_3)' = 1}$
$\mathbf{(f_1)'' = 2}$	$\mathbf{(f_2)'' = e^x}$	$\mathbf{(f_3)'' = 0}$

$$\mathbf{W(x)} = \begin{bmatrix} (f1) & (f2) & (f3) \\ (f1)' & (f2)' & (f3)' \\ (f1)'' & (f2)'' & (f3)'' \end{bmatrix}$$

$$\mathbf{detW(x)} = \begin{bmatrix} (f1) & (f2) & (f3) \\ (f1)' & (f2)' & (f3)' \\ (f1)'' & (f2)'' & (f3)'' \end{bmatrix} = \mathbf{0 + 2e^x + (2x^2e^x + 4x\,e^x) - (2xe^x + 4e^x) - 0 - (x^2e^x)}$$

detW(x) = $2xe^x - 2e^x + x^2e^x \neq 0$ for all x in the interval $(-\infty,+\infty)$, therefore the set is linear independent.

Question 4 (30 points)

Part A: If each vector is a linear combination of u, v, and w express it in terms of u, v, and w.

u = (1,2,3), **v** = (0,-1,5), and **w** = (-3,1,2)

(a) (0,2,-30)
(b) (2,5,7)
(c) (0,-2,20)
(d) (12,-8,-24)
(e) (4,0,50)
(f) (0,-2,30)
(g) (12,-9,-48)
(h) $\left(2,-\frac{4}{3},-4\right)$
(i) $\left(0,\frac{1}{9},-\frac{5}{3}\right)$

Solution:

(a) u + v – w (The negative sign may be place in front of any one vector and the answer is still correct)
(b) Not a linear combination.
(c) 2v + w or v + 2w
(d) -4u + w or 4u – w
(e) Not a linear combination
(f) u + (1/5)v + 5w
(g) Not a linear combination
(h) $\frac{1}{3}$**u – 2w**
(i) $\frac{1}{9}$**u + 2v -** $\frac{1}{4}$**w**

1. Assume u and v are vectors in R^n with the Euclidean inner product. Prove $\|u+v\|^2 = \|u\|^2 + \|v\|^2$. Justify each step. (30 points)

Solution:

	$\|u+v\|^2$	
=	$(u+v)\bullet(u+v)$	Definition of Euclidean length
=	$(u\bullet u)+(v\bullet v)+2(u\bullet v)$	Definition of inner product
=	$\|u\|^2+2(u\bullet v)+\|v\|^2$	Definition of Euclidean length
=	$\|u\|^2+\|v\|^2$	Definition of "orthogonal"

2. Find the standard matrix for the linear operator defined by the following equations and determine whether it's one-to-one. (40 points)

$w_1 = x_1 - 3x_2 - 2x_3$
$w_2 = 2x_1 + 5x_2$
$w_3 = 6x_1 + x_3$

Solution:

The standard matrix for the linear operator is:

$$\begin{bmatrix} 1 & -3 & -2 \\ 2 & 5 & 0 \\ 6 & 0 & 1 \end{bmatrix} = T_A$$

By the equivalent statements as defined in Chapter 4.3, T_A is one-to-one if $\det(A)\neq 0$. Find the determinant of T_A by cofactor expansion along the third row.

$$\begin{vmatrix} 1 & -3 & -2 \\ 2 & 5 & 0 \\ 6 & 0 & 1 \end{vmatrix} = 6\begin{vmatrix} -3 & -2 \\ 5 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & -3 \\ 2 & 5 \end{vmatrix} = 6(0+10) + 1(5+6) = 71 \neq 0.$$

Therefore, the linear operator shown above is one-to-one.

3. **a**. What is a linearly independent set? (5 pts.)
b. Show that the functions $\mathbf{f}_1 = \pi$, $\mathbf{f}_2 = (x/2)$, and $\mathbf{f}_3 = e^{-2x}$ form a linearly independent set of vectors on $C^2(-\infty,\infty)$. (25 pts.).

Solution:

a. A linearly independent set is formed when a nonempty set of vectors S= $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ has one and only one solution to the vector equation
$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$
namely,
$k_1 = 0, k_2 = 0, \ldots, k_r = 0.$

b. The Wronskian can be used to show that the functions form a linearly independent set of vectors if it does not equal zero. The form of the Wronskian is as follows:

$$W = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f'_1(x) & f'_2(x) & f'_3(x) \\ f''_1(x) & f''_2(x) & f''_3(x) \end{vmatrix}$$

So the Wronskian for the set of function is:

$$W = \begin{vmatrix} \pi & \frac{x}{2} & e^{-2x} \\ 0 & \frac{1}{2} & -2e^{-2x} \\ 0 & 0 & 4e^{-2x} \end{vmatrix}$$

This is found by multiplying the numbers on the main diagonal:

$W = \pi(1/2)\,(4\,e^{-2x}) = 2\,\pi\,e^{-2x} \neq 0$

Therefore, $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ form a linearly independent set.

4. For each of the following vectors
   i. Determine if the following form a basis for R^3
   ii. For those which do not form a basis for R^3, express each vector as a combination of the other 2 — not possible in general.
   iii. Define what a basis means..

   a. (0,5,4), (3,2,0), (0,9,-2)
   b. (6,2,1), (-4,8,9) (3,2,1)
   c. (2,1,2) (3,4,1) (-5.-10,1) (40 points)

Solution:

Note: for this problem, x=the first vector, y= the second and z= the third.

a. i.

$$\begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 9 & -2 & 0 \end{bmatrix} \quad \text{R1/3;R2/5;R3-9R2->R1} = \quad \begin{bmatrix} 1 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & \frac{4}{5} & 0 \\ 0 & 0 & \frac{-46}{5} & 0 \end{bmatrix}$$

The third line indicates that 0v1+0v2-46/5v3=0, which is false as v3 would be 0, thus the system is independent.

b. i.

form a basis (6,2,1)-(3,2,1)=(3,0,0)

c. i.

do not form a basis as 2x-3y=z

$$\begin{bmatrix} 2 & 1 & 2 \\ 3 & 4 & 1 \\ -5 & -10 & 1 \end{bmatrix} \quad \begin{matrix} \text{R2-R1=R2} \\ \text{R2<->R1} \\ \text{R3+R2->R3} \end{matrix} = \quad \begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 4 \\ 0 & 5 & -4 \end{bmatrix}$$

row 2=-row 3, is dependent

iii. A basis is a subset of vectors such that every vector in the larger set can be created from.

1. *Let* $V_1 = (1, 2, 1)$, $V_2 = (2, 9, 0)$, $V_3 = (3, 3, 4)$. *Show that the set* $S = \{V_1, V_2, V_3\}$ *is a basis* **for** R^3.

Solution :

To show that the set S spans R^3,

we must show that if there is cardinality between S and R^3 *then we need only prove that S is linearly independant or S spans* R^3 *to prove that S is a basis for* R^3 (not *both are needed*).

Since S contains 3 *vetors and it is in* R^3 *we know that there is cardinality between S and* R^3.

We know that if the determinant of the vectors in a matrix is not *zero then we know that S is linearly independant because its vectors are not linear combinations of the others.*

$$det(A) = \left| V_1 \;\; V_2 \;\; V_3 \right| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = -1$$

and so S is a basis for R^3.

2. Which of the following vectors are orthogonal?

a.)
(3, 5, 6, 0) and (4, 3, 3, 3)

b.)
(2, –2, 2, 2) and (5, 2, –6, 1)

c.)
(–2, 5, –1, 0) and (4, 3, 7, 3)

d.)
(1, 3, 0, 2) and (0, 3, 3, –5)

Solution:

a.) (3)(4) + (5)(3) + (6)(3) + (0)(3) = 45 = nonzero
no

b.) (2)(5) + (-2)(2) + (2)(-6) + (2)(1) = -4 = nonzero
no

c.) (-2)(4) + (5)(3) + (-1)(7) + (0)(3) = 0
yes

d.) (1)(0) + (3)(3) + (0)(3) + (2)(-5) = -1 = nonzero
no

3. *show that if* $\{v_1, v_2\}$ *is linearly independant and* v_3 *does not lie in the span* $\{v_1, v_2\}$, *then* $\{v_1, v_2, v_3\}$ *is linearly independant.*

Solution : By theorem 5.2.4 if v_3 *is not in the span of* $\{v_1, v_2\}$ *then it is* not *a linear combination of either* v_1 *nor* v_2. *Since there exists no linear combinations of* v_3 *in* $\{v_1, v_2\}$ *we say that it is linearly independant by theorem* 5.3.1

what does this mean?

definition of span, not use of any theorem.

We have a theorem which says that no vector in a linearly independent set is in the span of (or is a linear combination of) the remaining vectors. Moreover, this property is sufficient for the set to be linearly independent. From the first of these 2 facts we have that neither v_1 or v_2 is a linear combination of the other. From the second, and the stated property of v_3, we then have that $\{v_1, v_2, v_3\}$ is linearly independent

Given

$$A := \begin{pmatrix} 2 & 1 & 8 \\ 7 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix} \qquad u := \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \qquad v := \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Calculate
~~Solve for~~ $Au \cdot v$ and $u \cdot A^T v$.

Solution

$$Au = \begin{pmatrix} 2 & 1 & 8 \\ 7 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 29 \\ 21 \\ 10 \end{pmatrix} \qquad Au \cdot v = \begin{pmatrix} 29 \\ 21 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = 168$$

$$A^T = \begin{pmatrix} 2 & 7 & 0 \\ 1 & 1 & 1 \\ 8 & 2 & 3 \end{pmatrix} \qquad A^T v = \begin{pmatrix} 2 & 7 & 0 \\ 1 & 1 & 1 \\ 8 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 22 \\ 7 \\ 39 \end{pmatrix} \qquad u \cdot A^T \cdot v = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 22 \\ 7 \\ 39 \end{pmatrix} = 168$$

Cool — for a very simple problem.