

Math 313 Midterm III KEY
Spring 2011
sections 001 and 002
Instructor: Scott Glasgow

Write your name very clearly on this exam booklet. In this booklet, write your mathematics clearly, legibly, in big fonts, and, most important, “have a point”, i.e., make your work logically and even pedagogically acceptable. (Other human beings not already understanding 313 should be able to learn from your exam.) To avoid excessive erasing, first put your ideas together on scratch paper, then commit the logically acceptable fraction of your scratchings to this exam booklet. More is not necessarily better: say what you mean and mean what you say.

Good Practice: Look over the whole exam before attempting any problem.

Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed. Also, I will not even speak of this exam to anyone else in *any* fashion, not even whether it was difficult or not. My signature below indicates I accept this obligation.

Signature:

(Exams without this signature will not be graded.)

1. Find the orthogonal projection of $\mathbf{u} = (6, 1, 5, 2)$ onto the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1 = (1, 2, 1, 2)$, $\mathbf{v}_2 = (2, 3, 2, 1)$ and $\mathbf{v}_3 = (1, 3, 3, 1)$. It may or may not be useful for you to know that

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}, \quad (1.1)$$

where

$$\mathbf{u}_1 = (1 \ 2 \ 0 \ 4), \mathbf{u}_2 = (2 \ 1 \ 0 \ -1), \mathbf{u}_3 = (2 \ -3 \ -7 \ 1). \quad (1.2)$$

15 points

Solution a

The projection $\mathbf{u}_{\parallel}$ is $\mathbf{u} = (6, 1, 5, 2)$ iff $\mathbf{u} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, otherwise it is the vector $\mathbf{u}_{\parallel} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ nearest $\mathbf{u}$. Actually, in either case, we have

$$\mathbf{u}_{\parallel} = A\mathbf{x} \quad (1.3)$$

where $\mathbf{x}$ is any solution of

$$A^T A\mathbf{x} = A^T \mathbf{u},^1 \quad (1.4)$$

and where

$$A = [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}. \quad (1.5)$$

We solve (1.4) by row reducing $[A^T A | A^T \mathbf{u}]$ after calculating $A^T A$ and $A^T \mathbf{u}$:

¹ The vector $\mathbf{u}_{\parallel} = A\mathbf{x}$ is unique even if $\mathbf{x}$ solving (1.4) isn't: Let $\mathbf{x}_1$ and $\mathbf{x}_2$ both solve (1.4). Then $\mathbf{d} = \mathbf{x}_1 - \mathbf{x}_2$ satisfies $A^T A\mathbf{d} = \mathbf{0}$. Thus $A\mathbf{d} \in \text{Nul}(A^T) = (\text{Col}A)^{\perp}$, the latter by theorem. On the other hand we certainly have $A\mathbf{d} \in \text{Col}A$, more or less by definition. Thus $A\mathbf{d} \in \text{Col}A \cap (\text{Col}A)^{\perp} = \{\mathbf{0}\}$, the latter theorem simply being that the only vector orthogonal to itself is $\mathbf{0}$. Thus, for $\mathbf{u}_{\parallel 1} := A\mathbf{x}_1$ and $\mathbf{u}_{\parallel 2} := A\mathbf{x}_2$, we get $\mathbf{u}_{\parallel 1} - \mathbf{u}_{\parallel 2} = A\mathbf{x}_1 - A\mathbf{x}_2 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{d} = \mathbf{0}$, i.e. $\mathbf{u}_{\parallel 1} = \mathbf{u}_{\parallel 2}$.

$$\begin{aligned}
A^T A &= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 12 & 12 \\ 12 & 18 & 18 \\ 12 & 18 & 20 \end{bmatrix}, \quad A^T \mathbf{u} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 17 \\ 27 \\ 26 \end{bmatrix}, \\
\left[A^T A \mid A^T \mathbf{u} \right] &= \left[\begin{array}{ccc|c} 10 & 12 & 12 & 17 \\ 12 & 18 & 18 & 27 \\ 12 & 18 & 20 & 26 \end{array} \right] \sim \left[\begin{array}{ccc|c} 10 & 12 & 12 & 17 \\ 2 & 6 & 6 & 10 \\ 2 & 6 & 8 & 9 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & 5 \\ 10 & 12 & 12 & 17 \\ 2 & 6 & 8 & 9 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & 5 \\ 0 & 18 & 18 & 33 \\ 0 & 0 & 2 & -1 \end{array} \right] \\
&\sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & 5 \\ 0 & 6 & 6 & 11 \\ 0 & 0 & 2 & -1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & 5 \\ 0 & 6 & 0 & 14 \\ 0 & 0 & 2 & -1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & 6 & 6 & 10 \\ 0 & 3 & 0 & 7 \\ 0 & 0 & 2 & -1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & 6 & 0 & 13 \\ 0 & 3 & 0 & 7 \\ 0 & 0 & 2 & -1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 0 & 3 & 0 & 7 \\ 0 & 0 & 2 & -1 \end{array} \right] \\
&\sim \left[\begin{array}{ccc|c} 6 & 0 & 0 & -3 \\ 0 & 6 & 0 & 14 \\ 0 & 0 & 6 & -3 \end{array} \right].
\end{aligned}$$

(1.6)

Thus

$$\mathbf{u}_{\parallel} = A\mathbf{x} = \frac{1}{6} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 14 \\ -3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 22 \\ 27 \\ 16 \\ 5 \end{bmatrix} = \begin{bmatrix} 11/3 \\ 9/2 \\ 8/3 \\ 5/6 \end{bmatrix}. \quad (1.7)$$

Solution b

Note from the hint that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for the space, so that, by the “Fourier/ orthogonal projection theorem”, for $\mathbf{u} = (6, 1, 5, 2)$, the orthogonal projection is

$$\begin{aligned}
\mathbf{u}_{\parallel} &= \frac{\mathbf{u}_1 \cdot \mathbf{u}}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{u}_2 \cdot \mathbf{u}}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 + \frac{\mathbf{u}_3 \cdot \mathbf{u}}{\mathbf{u}_3 \cdot \mathbf{u}_3} \mathbf{u}_3 \\
&= \frac{\begin{pmatrix} 1 & 2 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} 6 & 1 & 5 & 2 \end{pmatrix}}{\begin{pmatrix} 1 & 2 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 & 4 \end{pmatrix}} \begin{pmatrix} 1 & 2 & 0 & 4 \end{pmatrix} + \\
&\quad \frac{\begin{pmatrix} 2 & 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 6 & 1 & 5 & 2 \end{pmatrix}}{\begin{pmatrix} 2 & 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 0 & -1 \end{pmatrix}} \begin{pmatrix} 2 & 1 & 0 & -1 \end{pmatrix} + \\
&\quad \frac{\begin{pmatrix} 2 & -3 & -7 & 1 \end{pmatrix} \cdot \begin{pmatrix} 6 & 1 & 5 & 2 \end{pmatrix}}{\begin{pmatrix} 2 & -3 & -7 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 & -7 & 1 \end{pmatrix}} \begin{pmatrix} 2 & -3 & -7 & 1 \end{pmatrix} \\
&= \frac{16}{3 \cdot 7} \begin{pmatrix} 1 & 2 & 0 & 4 \end{pmatrix} + \frac{11}{2 \cdot 3} \begin{pmatrix} 2 & 1 & 0 & -1 \end{pmatrix} + \frac{-24}{3 \cdot 3 \cdot 7} \begin{pmatrix} 2 & -3 & -7 & 1 \end{pmatrix} \\
&= \frac{\left(2^5 \cdot 3 \cdot \begin{pmatrix} 1 & 2 & 0 & 4 \end{pmatrix} + 3 \cdot 7 \cdot 11 \begin{pmatrix} 2 & 1 & 0 & -1 \end{pmatrix} - 2^4 \cdot 3 \begin{pmatrix} 2 & -3 & -7 & 1 \end{pmatrix} \right)}{2 \cdot 3^2 \cdot 7} \\
&= \frac{\left(2^5 \cdot 3 + 2 \cdot 3 \cdot 7 \cdot 11 - 2^5 \cdot 3 \quad 2^6 \cdot 3 + 3 \cdot 7 \cdot 11 + 2^4 \cdot 3^2 \quad 2^4 \cdot 3 \cdot 7 \quad 2^7 \cdot 3 - 3 \cdot 7 \cdot 11 - 2^4 \cdot 3 \right)}{2 \cdot 3^2 \cdot 7} \\
&= \frac{\begin{pmatrix} 2 \cdot 11 & (2^4 + 11) & 2^4 & (2^4 - 11) \end{pmatrix}}{2 \cdot 3} = \frac{\begin{pmatrix} 2 \cdot 11 & 3^3 & 2^4 & 5 \end{pmatrix}}{2 \cdot 3} = \begin{pmatrix} \frac{11}{3} & \frac{3^2}{2} & \frac{2^3}{3} & \frac{5}{2 \cdot 3} \end{pmatrix}.
\end{aligned}$$

(1.8)

2. Use the Gram-Schmidt process, etc., to transform the following 3 members of the Euclidean space $\mathbb{R}^4$ into an orthonormal set in that innerproduct space:

$$\mathbf{u}_1 = (1, 2, 1, 2), \mathbf{u}_2 = (2, 3, 2, 1), \text{ and } \mathbf{u}_3 = (1, 3, 3, 1). \quad (1.9)$$

It may or may not be useful for you to know that

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad (1.10)$$

15 points

Solution

We could produce an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for 3-dimensional subspace

$W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^4$ from the original basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of that subspace via (a modified version of) Gram-Schmidt: we first row reduce a certain matrix to give the fact stated in the hint...

Row Reduction:

$$\begin{aligned}
 A^T &:= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -1 & 0 & -3 \\ 0 & 1 & 2 & -1 \end{bmatrix} \xrightarrow{R_3 + R_2} \\
 &\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 2 & -4 \end{bmatrix} \xrightarrow[1/2 * R_3]{-1 * R_2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 - R_3} \\
 &\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} =: \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{bmatrix} \dots
 \end{aligned} \tag{1.11}$$

and then perform G.S. on the already-simplified basis:

Gram-Schmidt:

$$\begin{aligned}
 \mathbf{v}_1 &:= \mathbf{w}_1 = (1 \ 2 \ 0 \ 4), \\
 \mathbf{v}_2 &\propto \mathbf{w}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \\
 &= (0 \ 1 \ 0 \ 3) - \frac{(1 \ 2 \ 0 \ 4) \cdot (0 \ 1 \ 0 \ 3)}{(1 \ 2 \ 0 \ 4) \cdot (1 \ 2 \ 0 \ 4)} (1 \ 2 \ 0 \ 4) \\
 &= (0 \ 1 \ 0 \ 3) - \frac{14}{21} (1 \ 2 \ 0 \ 4) \\
 &= (0 \ 1 \ 0 \ 3) - \frac{2}{3} (1 \ 2 \ 0 \ 4) \\
 &\propto 3(0 \ 1 \ 0 \ 3) - 2(1 \ 2 \ 0 \ 4) \\
 &= (0 \ 3 \ 0 \ 9) - (2 \ 4 \ 0 \ 8) \propto (2 \ 1 \ 0 \ -1) =: \mathbf{v}_2 \\
 \mathbf{v}_3 &\propto \mathbf{w}_3 - \frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\
 &= (0 \ 0 \ 1 \ -2) - \frac{(1 \ 2 \ 0 \ 4) \cdot (0 \ 0 \ 1 \ -2)}{(1 \ 2 \ 0 \ 4) \cdot (1 \ 2 \ 0 \ 4)} (1 \ 2 \ 0 \ 4) \\
 &\quad - \frac{(2 \ 1 \ 0 \ -1) \cdot (0 \ 0 \ 1 \ -2)}{(2 \ 1 \ 0 \ -1) \cdot (2 \ 1 \ 0 \ -1)} (2 \ 1 \ 0 \ -1) \\
 &= (0 \ 0 \ 1 \ -2) + \frac{8}{21} (1 \ 2 \ 0 \ 4) - \frac{1}{3} (2 \ 1 \ 0 \ -1) \\
 &\propto 21(0 \ 0 \ 1 \ -2) + 8(1 \ 2 \ 0 \ 4) - 7(2 \ 1 \ 0 \ -1) \\
 &= (-6 \ 9 \ 21 \ -3) \propto (2 \ -3 \ -7 \ 1) =: \mathbf{v}_3.
 \end{aligned} \tag{1.12}$$

Finally we normalize each vector to obtain the required orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$:

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1 \ 2 \ 0 \ 4)}{\sqrt{21}}, \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(2 \ 1 \ 0 \ -1)}{\sqrt{6}}, \quad \mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{(2 \ -3 \ -7 \ 1)}{\sqrt{63}}. \tag{1.13}$$

3. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for an innerproduct space V , then, for any vector $\mathbf{v} \in V$,

$$\mathbf{v} = \frac{\langle \mathbf{v}_1, \mathbf{v} \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{v}_2, \mathbf{v} \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 + \dots + \frac{\langle \mathbf{v}_n, \mathbf{v} \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle} \mathbf{v}_n. \tag{1.14}$$

15 points

Solution

By previous theorem, since $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V , we immediately have that for any $\mathbf{v} \in V$ there exists one and only one coordinate vector $(c_1, c_2, \dots, c_n) \in \mathbb{R}^n$ such that

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n. \quad (1.15)$$

Taking the innerproduct of both sides of (1.15) with $\mathbf{v}_j \in \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n\}$, we get

$$\begin{aligned} \langle \mathbf{v}_j, \mathbf{v} \rangle &= \langle \mathbf{v}_j, c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_j \mathbf{v}_j + \dots + c_n \mathbf{v}_n \rangle \\ &= c_1 \langle \mathbf{v}_j, \mathbf{v}_1 \rangle + c_2 \langle \mathbf{v}_j, \mathbf{v}_2 \rangle + \dots + c_j \langle \mathbf{v}_j, \mathbf{v}_j \rangle + \dots + c_n \langle \mathbf{v}_j, \mathbf{v}_n \rangle \\ &= 0 + 0 + \dots + c_j \langle \mathbf{v}_j, \mathbf{v}_j \rangle + \dots + 0 = c_j \langle \mathbf{v}_j, \mathbf{v}_j \rangle, \end{aligned} \quad (1.16)$$

so that (1.14) follows (from (1.16) injected into (1.15)). (Note that since $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis it is linearly independent, hence none of its elements are the additive identity, hence $\langle \mathbf{v}_j, \mathbf{v}_j \rangle \neq 0$ for all j .)

4. Find the angle $\theta \in [0, \pi]$ between $f(x) = 2x - 1$ and $g(x) = 6x^2 - 6x + 1$ in the innerproduct space $P_2 := \text{Span}\{1, x, x^2\}$, given the inner product

$$\langle f, g \rangle := \int_0^1 f(x)g(x)dx. \quad (1.17)$$

What is the angle $\theta \in [0, \pi]$ between $f(x) = 2x - 1$ and $g(x) = 6x^2 - 6x + 1$ when the inner product is changed to

$$\langle f, g \rangle := \int_{-1}^1 f(x)g(x)dx ? \quad (1.18)$$

Actually, in this latter case, just tell me whether $\theta \in [0, \pi/2)$, or $\theta = \pi/2$ or $\theta \in (\pi/2, \pi]$.

15 points

Solution

By (good) definition,

$$\theta = \cos^{-1} \left(\frac{\langle f, g \rangle}{\|f\| \|g\|} \right) = \cos^{-1} \left(\frac{\langle f, g \rangle}{\sqrt{\langle f, f \rangle} \sqrt{\langle g, g \rangle}} \right). \quad (1.19)$$

But since

$$\begin{aligned} \langle f, g \rangle &= \int_0^1 f(x)g(x)dx = \int_0^1 (2x-1)(6x^2-6x+1)dx = \int_0^1 (2x-1)d(2x^3-3x^2+x) \\ &= (2x-1)(2x^3-3x^2+x) \Big|_{x=0}^{x=1} - \int_0^1 (2x^3-3x^2+x)d(2x-1) \\ &= 1 \cdot (2-3+1) - (x^4-2x^3+x^2) \Big|_{x=0}^{x=1} = 0 - (1-2+1) = 0, \\ \langle f, f \rangle &:= \int_0^1 f(x)f(x)dx = \int_0^1 f^2(x)dx = \int_0^1 (2x-1)^2 dx = \frac{(2x-1)^3}{6} \Big|_{x=0}^{x=1} \\ &= \frac{1^3}{6} - \frac{(-1)^3}{6} = \frac{1}{3} \neq 0, \\ \langle g, g \rangle &:= \int_0^1 g(x)g(x)dx = \int_0^1 (6x^2-6x+1)(6x^2-6x+1)dx \\ &= \int_0^1 (6x^2-6x+1)d(2x^3-3x^2+x) = (6x^2-6x+1)(2x^3-3x^2+x) \Big|_{x=0}^{x=1} \\ &\quad - \int_0^1 (2x^3-3x^2+x)d(6x^2-6x+1) \\ &= 0 - 6 \int_0^1 (2x^3-3x^2+x)(2x-1)dx = -3 \int_0^1 (2x-1)d(x^4-2x^3+x^2) \\ &= -3 \left((2x-1)(x^4-2x^3+x^2) \Big|_{x=0}^{x=1} - \int_0^1 (x^4-2x^3+x^2)d(2x-1) \right) \\ &= 3 \int_0^1 (x^4-2x^3+x^2)d(2x-1) = 6 \left[\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 \right]_{x=0}^{x=1} \\ &= \frac{1}{5}(6-15+10) = \frac{1}{5} \neq 0, \end{aligned} \quad (1.20)$$

clearly

$$\theta = \cos^{-1} \left(\frac{\langle f, g \rangle}{\|f\| \|g\|} \right) = \cos^{-1} \left(\frac{0}{\sqrt{\langle f, f \rangle} \sqrt{\langle g, g \rangle}} \right) = \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ. \quad (1.21)$$

For the new innerproduct (1.18) we get

$$\begin{aligned} \langle f, g \rangle &= \int_{-1}^1 f(x)g(x)dx = \int_{-1}^1 (2x-1)(6x^2-6x+1)dx = \int_{-1}^1 (2x-1)d(2x^3-3x^2+x) \\ &= (2x-1)(2x^3-3x^2+x) \Big|_{x=-1}^{x=1} - \int_{-1}^1 (2x^3-3x^2+x)d(2x-1) \\ &= 1 \cdot (2-3+1) - (-3)(-2-3-1) - (x^4-2x^3+x^2) \Big|_{x=-1}^{x=1} = -18+4 = -14, \\ \langle f, f \rangle &:= \int_{-1}^1 f(x)f(x)dx = \int_{-1}^1 f^2(x)dx = \int_{-1}^1 (2x-1)^2 dx = \frac{(2x-1)^3}{6} \Big|_{x=-1}^{x=1} \\ &= \frac{1^3}{6} - \frac{(-3)^3}{6} = \frac{1}{6} + \frac{27}{6} = \frac{28}{6} = \frac{14}{3} \neq 0, \\ \langle g, g \rangle &:= \int_{-1}^1 g(x)g(x)dx = \int_{-1}^1 (6x^2-6x+1)(6x^2-6x+1)dx \\ &= \int_{-1}^1 (6x^2-6x+1)d(2x^3-3x^2+x) = (6x^2-6x+1)(2x^3-3x^2+x) \Big|_{x=-1}^{x=1} \\ &\quad - \int_{-1}^1 (2x^3-3x^2+x)d(6x^2-6x+1) \\ &= 0 - (6+6+1)(-2-3-1) - 6 \int_{-1}^1 (2x^3-3x^2+x)(2x-1)dx = 2 \cdot 3 \cdot 13 - 3 \int_{-1}^1 (2x-1)d(x^4-2x^3+x^2) \\ &= 2 \cdot 3 \cdot 13 - 3 \left((2x-1)(x^4-2x^3+x^2) \Big|_{x=-1}^{x=1} - \int_{-1}^1 (x^4-2x^3+x^2)d(2x-1) \right) \\ &= 2 \cdot 3 \cdot 13 - 3^2 \cdot 4 + 3 \int_{-1}^1 (x^4-2x^3+x^2)d(2x-1) = 42 + 12 \left[\frac{1}{5}x^5 + \frac{1}{3}x^3 \right]_{x=0}^{x=1} = 48 \frac{2}{5}, \end{aligned} \quad (1.22)$$

so that

$$\begin{aligned}
\theta &= \cos^{-1} \left(\frac{\langle f, g \rangle}{\|f\| \|g\|} \right) = \cos^{-1} \left(\frac{-14}{\sqrt{\frac{14}{3} \cdot \frac{242}{5}}} \right) = \cos^{-1} \left(-\sqrt{\frac{3 \cdot 5 \cdot 7}{11 \cdot 11}} \right) \\
&= \cos^{-1} \left(-\sqrt{\frac{105}{121}} \right) \in \left(\frac{\pi}{2}, \pi \right].
\end{aligned} \tag{1.23}$$

5. Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}. \tag{1.24}$$

What is the diagonalized form of A associated with your specific P ? Give a formula for A^n , where n is, say, a positive integer. It may or may not be helpful to note that

$$\begin{bmatrix} -1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{bmatrix} \tag{1.25}$$

15 points

Solution

We seek an invertible matrix P with eigenvectors of A as its columns. First we calculate A 's eigenvalues:

λ is an eigenvalue of A iff

$$\begin{aligned}
 0 &= \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} = -\lambda(2-\lambda)(3-\lambda) - 2(0-1(2-\lambda)) \\
 &= (2-\lambda)(-\lambda(3-\lambda)+2) = (2-\lambda)(\lambda^2-3\lambda+2) = (2-\lambda)(\lambda-2)(\lambda-1) \quad (1.26) \\
 &\Leftrightarrow \\
 &\lambda = 1, 2, 2.
 \end{aligned}$$

The associated eigenvectors are elements of the nontrivial nullspaces of $A - \lambda I$:

$$\begin{aligned}
 \mathbf{0} \neq \mathbf{x}_{\lambda=1} \in \text{Nul}(A - 1 \cdot I) &= \text{Nul} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\
 &= \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad (1.27)
 \end{aligned}$$

$$\mathbf{0} \neq \mathbf{x}_{\lambda=2} \in \text{Nul}(A - 2 \cdot I) = \text{Nul} \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Thus

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad (1.28)$$

diagonalizes A to

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (1.29)$$

Moreover, from (1.25) we find that

$$P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}, \quad (1.30)$$

so that since

$$A = PDP^{-1} \quad (1.31)$$

we get, theorem,

$$\begin{aligned} A^n &= (PDP^{-1})^n = PD^nP^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -2^n & 0 & -2 \\ 0 & 2^n & 1 \\ 2^n & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \quad (1.32) \\ &= \begin{bmatrix} -2^n + 2 & 0 & -2^{n+1} + 2 \\ 2^n - 1 & 2^n & 2^n - 1 \\ 2^n - 1 & 0 & 2^{n+1} - 1 \end{bmatrix}. \end{aligned}$$

6. Let A be an $n \times n$ matrix. Prove that if

$$\mathbf{Ax} \cdot \mathbf{Ay} = \mathbf{x} \cdot \mathbf{y} \quad (1.33)$$

holds for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then A is orthogonal.

15 points

Solution

$$\begin{aligned} \mathbf{x}^T I \mathbf{y} &= \mathbf{x}^T \mathbf{y} = \mathbf{x} \cdot \mathbf{y} = \mathbf{Ax} \cdot \mathbf{Ay} = (\mathbf{Ax})^T \mathbf{Ay} = \mathbf{x}^T A^T \mathbf{Ay} \\ &\Leftrightarrow \\ 0 &= \mathbf{x}^T A^T \mathbf{Ay} - \mathbf{x}^T I \mathbf{y} = \mathbf{x}^T (A^T A - I) \mathbf{y}. \end{aligned} \quad (1.34)$$

Since (1.34) holds for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, it holds for $\mathbf{x} = (A^T A - I) \mathbf{y}$, so that (1.34) becomes

$$\begin{aligned} 0 &= ((A^T A - I) \mathbf{y})^T (A^T A - I) \mathbf{y} = \|(A^T A - I) \mathbf{y}\|^2 \\ &\Leftrightarrow \\ \mathbf{0} &= (A^T A - I) \mathbf{y} \end{aligned} \quad (1.35)$$

for every $\mathbf{y} \in \mathbb{R}^n$. Thus (1.35) holds for $\mathbf{y}$ chosen as each of the rows of the matrix $A^T A - I$, showing that each of these rows is the zero vector (why?) and, so, showing that $A^T A - I$ is the zero matrix, i.e. $A^T A = I$, which, together with A square, means that A is an orthogonal matrix.

7. Prove that if A is an orthogonal $n \times n$ matrix, then

$$\|A\mathbf{x}\| = \|\mathbf{x}\| \quad (1.36)$$

for every $\mathbf{x} \in \mathbb{R}^n$.

15 points

Solution

For every $\mathbf{x} \in \mathbb{R}^n$ we have

$$\begin{aligned} \|A\mathbf{x}\|^2 &= (A\mathbf{x})^T A\mathbf{x} = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T I\mathbf{x} = \mathbf{x}^T \mathbf{x} = \|\mathbf{x}\|^2 \\ &\Leftrightarrow \\ \|A\mathbf{x}\| &= \|\mathbf{x}\|. \end{aligned} \quad (1.37)$$

8. Find the least squares solution $\mathbf{x} \in \mathbb{R}^2$ to the following (inconsistent?) system of equations:

$$A\mathbf{x} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{b}. \quad (1.38)$$

15 points

Solution

The point value is small, because the matrix A has already been factored into QR , with Q containing orthonormal columns (but not said to be “orthogonal”, as per the parlance, since it is not square), and with R upper triangular and invertible (which, in this

construction, implies, and is implied by, A 's columns being independent). Thus, as we have discussed, the (possibly inconsistent) system $A\mathbf{x} = Q\mathbf{R}\mathbf{x} = \mathbf{b}$ gives rise to the following form for the normal equations:

$$\begin{aligned} R^T R\mathbf{x} &= R^T I R\mathbf{x} = R^T Q^T Q R\mathbf{x} = (QR)^T Q R\mathbf{x} = A^T A\mathbf{x} = A^T \mathbf{b} = R^T Q^T \mathbf{b} \\ &\Leftrightarrow \\ R\mathbf{x} &= Q^T \mathbf{b}. \end{aligned} \quad (1.39)$$

the latter since R invertible implies R^T invertible. Thus the possibly inconsistent system (1.38) gives rise to the following system which a) is guaranteed to be consistent and which b) gives the least square solution of (1.38):

$$R\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = Q^T \mathbf{b}. \quad (1.40)$$

So then if we write $\mathbf{x} = (x_1, x_2)^T$, clearly (using back substitution)

$$x_2 = -1, \quad x_1 = 1 - 1 \cdot x_2 = 1 - 1 \cdot (-1) = 2. \quad (1.41)$$

9. Find the linear function $l(x) = \alpha + \beta x$ that is closest to the quadratic polynomial $q(x) = 1 + 6x^2$, given that the notion of distance is given by

$$(d(f_1, f_2))^2 := \|f_1 - f_2\|^2 := \langle f_1 - f_2, f_1 - f_2 \rangle, \quad (1.42)$$

where

$$\langle f_1, f_2 \rangle = \int_0^1 f_1(x) f_2(x) dx. \quad (1.43)$$

Use Gram-Schmidt to generate an appropriate orthogonal basis, and the relevant extension of the orthogonal basis representation theorem (that we have termed “Fourier’s theorem”), to find $l(x)$.

You may find it convenient to know that

$$\begin{aligned}
\int_0^1 1 \cdot 1 dx &= 1, \int_0^1 1 \cdot x dx = \frac{1}{2}, \int_0^1 1 \cdot (1+6x^2) dx = 3, \int_0^1 1 \cdot (2x-1) dx = 0, \int_0^1 x \cdot (2x-1) dx = \frac{1}{6}, \\
\int_0^1 x \cdot x dx &= \frac{1}{3}, \int_0^1 x \cdot (1+6x^2) dx = 2, \int_0^1 (2x-1) \cdot (2x-1) dx = \frac{1}{3}, \int_0^1 (2x-1) \cdot (1+6x^2) dx = 1, \\
\int_0^1 (1+6x^2) \cdot (1+6x^2) dx &= \frac{61}{5}.
\end{aligned}
\tag{1.44}$$

20 points

Solution

A “big”/ambient inner product space V in which both $q(x)$ and $l(x)$ live is $P_2 = \text{Span}\{1, x, x^2\}$, the inner product as in (1.43). “Fourier’s theorem” is that if $\{\mathbf{v}_1, \dots, \mathbf{v}_r, \mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V , then

$$\mathbf{u}_{\parallel} := \frac{\langle \mathbf{v}_1, \mathbf{u} \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \dots + \frac{\langle \mathbf{v}_r, \mathbf{u} \rangle}{\langle \mathbf{v}_r, \mathbf{v}_r \rangle} \mathbf{v}_r + 0\mathbf{v}_{r+1} + \dots + 0\mathbf{v}_n = \frac{\langle \mathbf{v}_1, \mathbf{u} \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \dots + \frac{\langle \mathbf{v}_r, \mathbf{u} \rangle}{\langle \mathbf{v}_r, \mathbf{v}_r \rangle} \mathbf{v}_r \tag{1.45}$$

is the element of

$$\begin{aligned}
W &= \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_r, 0\mathbf{v}_{r+1}, \dots, 0\mathbf{v}_n\} = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\} \\
&\subset_{\text{space}} \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_r, \mathbf{v}_{r+1}, \dots, \mathbf{v}_n\} = V
\end{aligned}
\tag{1.46}$$

that is nearest $\mathbf{u} \in V$. Now the linear function $l(x) = \alpha + \beta x$ is in

$$W = \text{Span}\{1, x, 0x^2\} = \text{Span}\{1, x\} \subset_{\text{space}} \text{Span}\{1, x, x^2\} = V = P_2, \tag{1.47}$$

so it is enough to perform Gram-Schmidt on the basis $B = \{1, x\} =: \{\mathbf{u}_1, \mathbf{u}_2\}$ for W to use formula (1.45). We get an orthogonal basis $B_{\perp} = \{\mathbf{v}_1, \mathbf{v}_2\}$ via (the modified G-S algorithm)

$$\mathbf{v}_1 = \mathbf{u}_1 = 1, \mathbf{v}_2 \propto \mathbf{u}_2 - \frac{\langle \mathbf{v}_1, \mathbf{u}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 = x - \frac{\langle 1, x \rangle}{\langle 1, 1 \rangle} \cdot 1 = x - \frac{\int_0^1 1 \cdot x dx}{\int_0^1 1 \cdot 1 dx} \cdot 1 = x - \frac{\frac{1}{2}}{1} \cdot 1 \propto 2x - 1 = \mathbf{v}_2. \tag{1.48}$$

Thus (1.45) is

$$\begin{aligned}
(1+6x^2)_{||} &= \frac{\langle \mathbf{v}_1, 1+6x^2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{v}_2, 1+6x^2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 \\
&= \frac{\langle 1, 1+6x^2 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle 2x-1, 1+6x^2 \rangle}{\langle 2x-1, 2x-1 \rangle} (2x-1) \\
&= \frac{\int_0^1 1 \cdot (1+6x^2) dx}{\int_0^1 1 \cdot 1 dx} + \frac{\int_0^1 (2x-1) \cdot (1+6x^2) dx}{\int_0^1 (2x-1) \cdot (2x-1) dx} (2x-1) \\
&= \frac{3}{1} + \frac{1}{\frac{1}{3}} (2x-1) = 6x.
\end{aligned} \tag{1.49}$$

10. Determine whether the proposed innerproduct below is actually an inner product, i.e., determine whether it satisfies the 3 or so axioms defining an innerproduct.

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle (u_1, u_2), (v_1, v_2) \rangle := u_1 v_1 + 3u_2 v_1 + 3u_1 v_2 + u_2 v_2. \tag{1.50}$$

(Show that each axiom does or does not hold.)

15 points

Solution

Clearly the innerproduct passes all tests/axioms except possibly positivity: it is clearly symmetric, and linear in each argument. To check positivity=nonnegativity+nondegeneracy we complete the square: definition (1.50) gives

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{u} \rangle &= \langle (u_1, u_2), (u_1, u_2) \rangle := u_1 u_1 + 3u_2 u_1 + 3u_1 u_2 + u_2 u_2 = u_1^2 + 6u_1 u_2 + u_2^2 \\
&= (u_1 + 3u_2)^2 - 9u_2^2 + u_2^2 = (u_1 + 3u_2)^2 - 8u_2^2 = (u_1 + \sqrt{9}u_2 + \sqrt{8}u_2)(u_1 + \sqrt{9}u_2 - \sqrt{8}u_2) \\
&= (u_1 + (\sqrt{9} + \sqrt{8})u_2)(u_1 + (\sqrt{9} - \sqrt{8})u_2).
\end{aligned} \tag{1.51}$$

This form is clearly not nonnegative, so clearly “highly degenerate”: (1.51) implies for example that

$$\langle \mathbf{u}, \mathbf{u} \rangle = \langle (u_1, u_2), (u_1, u_2) \rangle = \langle (-3u_2, u_2), (-3u_2, u_2) \rangle = (-3u_2 + 3u_2)^2 - 8u_2^2 = -8u_2^2 \leq 0, \tag{1.52}$$

with strict inequality usually, and that

$$\langle \mathbf{u}, \mathbf{u} \rangle = 0, \quad (1.53)$$

for $\mathbf{u}$ in the union of two 1-dimensional subspaces, namely

$$\mathbf{u} \in \text{span} \left\{ \begin{pmatrix} \sqrt{9} + \sqrt{8} \\ -1 \end{pmatrix} \right\} \cup \text{span} \left\{ \begin{pmatrix} \sqrt{9} - \sqrt{8} \\ -1 \end{pmatrix} \right\}. \quad (1.54)$$

Thus this proposed innerproduct is only a “poser”, and in fact it is not even close to an (admissible) innerproduct.