

Math 313 Midterm II KEY
Spring 2011
sections 001 and 002
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Write your name very clearly on this exam booklet. In this booklet, write your mathematics clearly, legibly, in big fonts, and, most important, “have a point”, i.e., make your work logically and even pedagogically acceptable. (Other human beings not already understanding 313 should be able to learn from your exam.) To avoid excessive erasing, first put your ideas together on scratch paper, then commit the logically acceptable fraction of your scratchings to this exam booklet. More is not necessarily better: say what you mean and mean what you say.

Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed. Also, I will not even speak of this exam to anyone else in *any* fashion, not even whether it was difficult or not. My signature below indicates I accept this obligation.

Signature:

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- 1) For the given set of objects, together with the indicated notions of addition and scalar multiplication, determine whether each of the ten vector space axioms holds: real pairs (x, y) , where

$$(x, y) + (x', y') := (x + x', y + y'), \quad k(x, y) := \left(\frac{kx + 2ky}{5}, \frac{2kx + 4ky}{5} \right). \quad (1)$$

25pts

Solution

- 1) through 5) : Since $V = \mathbb{R}^2$ but with only scalar multiplication differing, these axioms hold (since they reference only vector addition).

- 6) $k(x, y) \in V$ when $(x, y) \in V$ and $k \in \mathbb{R}$ since both $\frac{kx + 2ky}{5}$ and $\frac{2kx + 4ky}{5}$ are clearly real numbers then.

- 7) We have

$$\begin{aligned} k((x, y) + (w, z)) &= k(x + w, y + z) = \left(\frac{k(x + w) + 2k(y + z)}{5}, \frac{2k(x + w) + 4k(y + z)}{5} \right) \\ &= \left(\frac{kx + 2ky}{5} + \frac{kx + 2ky}{5}, \frac{2kx + 4ky}{5} + \frac{2kx + 4ky}{5} \right) \\ &= \left(\frac{kx + 2ky}{5}, \frac{2kx + 4ky}{5} \right) + \left(\frac{kx + 2ky}{5}, \frac{2kx + 4ky}{5} \right) \\ &= k(x, y) + k(w, z), \end{aligned}$$

so this axiom holds.

- 8) We have

$$\begin{aligned} (k + m)(x, y) &= \left(\frac{(k + m)x + 2(k + m)y}{5}, \frac{2(k + m)x + 4(k + m)y}{5} \right) \\ &= \left(\frac{kx + 2ky}{5} + \frac{mx + 2my}{5}, \frac{2kx + 4ky}{5} + \frac{2mx + 4my}{5} \right) \\ &= \left(\frac{kx + 2ky}{5}, \frac{2kx + 4ky}{5} \right) + \left(\frac{mx + 2my}{5}, \frac{2mx + 4my}{5} \right) \\ &= k(x, y) + m(x, y), \end{aligned}$$

so this axiom holds

- 9) We have

$$\begin{aligned}
k(m(x, y)) &= k\left(\frac{mx + 2my}{5}, \frac{2mx + 4my}{5}\right) \\
&= \left(\frac{k\left(\frac{mx + 2my}{5}\right) + 2k\left(\frac{2mx + 4my}{5}\right)}{5}, \frac{2k\left(\frac{mx + 2my}{5}\right) + 4k\left(\frac{2mx + 4my}{5}\right)}{5} \right) \quad (2) \\
&= \left(\frac{\frac{5kmx}{5} + \frac{2 \cdot 5kmy}{5}}{5}, \frac{\frac{2 \cdot 5kmx}{5} + \frac{4 \cdot 5kmy}{5}}{5} \right) \\
&= \left(\frac{(km)x + 2(km)y}{5}, \frac{2(km)x + 4(km)y}{5} \right) = (km)(x, y),
\end{aligned}$$

so this axiom holds.

10) We have

$$1(x, y) := \left(\frac{1 \cdot x + 2 \cdot 1 \cdot y}{5}, \frac{2 \cdot 1 \cdot x + 4 \cdot 1 \cdot y}{5} \right) = \left(\frac{x + 2y}{5}, \frac{2x + 4y}{5} \right)$$

which is not (x, y) in every instance. For example,

$$1(2, -1) = \left(\frac{2 + 2 \cdot (-1)}{5}, \frac{2 \cdot 2 + 4 \cdot (-1)}{5} \right) = (0, 0) \neq (2, -1). \text{ Thus all axioms hold except the last.}$$

2) Prove that for any (real) vector space $(V, \mathbb{R}, +, \cdot)$ (satisfying the ten axioms)—no matter how bizarre the addition $+$ and the scalar multiplication $\cdot$ are—we must have $0 \cdot \mathbf{u} = \mathbf{z}$ for any vector $\mathbf{u} \in V$ ($\in \mathbb{R}$), where $\mathbf{z}$ is the “zero” vector in the space, i.e. where $\mathbf{z}$ is the additive identity in V . Be sure to list the axioms used in your proof. Feel free to use the fact that

$$\begin{aligned}
\mathbf{w} + \mathbf{v} = \mathbf{v} &\Rightarrow \mathbf{w} = \mathbf{z}, \text{ or} \\
\mathbf{v} + \mathbf{w} = \mathbf{v} &\Rightarrow \mathbf{w} = \mathbf{z},
\end{aligned} \quad (3)$$

i.e. feel free to use the fact that if a vector $\mathbf{w}$ acts like $\mathbf{z}$ even for just one $\mathbf{v} \in V$, then it is $\mathbf{z}$. On the other hand, you may also do what you did in the relevant type of homework problems (which invents the fact indicated in equation (3) for you).

10pts

Solution

By axiom 8)

$$0 \bullet \mathbf{u} + 0 \bullet \mathbf{u} = (0 + 0) \bullet \mathbf{u}, \quad (4)$$

which, by property of the number $0 \in \mathbb{R}$, gives

$$0 \bullet \mathbf{u} + 0 \bullet \mathbf{u} = 0 \bullet \mathbf{u}. \quad (5)$$

But now this is the left-hand side of equation (3) above with $\mathbf{w} = 0 \bullet \mathbf{u}$ (and, less important, $\mathbf{v} = 0 \bullet \mathbf{u}$). So by the right-hand side of equation (3) $\mathbf{w} = 0 \bullet \mathbf{u} = \mathbf{z}$.

- 3) By use of the relevant “if and only if” theorem, determine whether the following is a subspace of M_{nn} : (M_{nn} is the vector space of $n \times n$ matrices with ordinary matrix addition and scalar multiplication.) the set of all $n \times n$ matrices A such that $AB = BA$ for some fixed/specific/particular $n \times n$ matrix B . MAKE SURE AND REFERENCE AND **USE** THE THEOREM in determining your conclusion.

15pts

Solution

Since such A 's certainly form a non-empty subset of M_{nn} (choose $A = B$ for example), then, theorem, they form a subspace of M_{nn} iff for all scalars c and c' and for every A and A' in the set we have

$$(cA + c'A')B = B(cA + c'A'). \quad (6)$$

Now (6) always holds and, so, the set is actually a subspace of M_{nn} : we have

$$(cA + c'A')B = (c(AB) + c'(A'B)) = (c(BA) + c'(BA')) = B(cA + c'A') \quad (7)$$

by matrix algebra, together with A and A' 's membership in the set (giving both $AB = BA$ and $A'B = BA'$).

4) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V . It turns out that

$S' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a basis for V , where

$$\mathbf{u}_1 = \mathbf{v}_1, \quad \mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{u}_3 = \mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3. \quad (8)$$

So suppose the coordinate vector $(\mathbf{v})_S$ of a vector $\mathbf{v} \in V$ relative to basis S is given by

$$(\mathbf{v})_S = (a, b, c). \quad (9)$$

What is the coordinate vector $(\mathbf{v})_{S'}$ of $\mathbf{v}$ relative to basis S' ?

15pts

Solution

By definition of coordinate vector, (9) holds with $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ iff

$$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3. \quad (10)$$

(8) is inverted by noting then that

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1, \quad \mathbf{v}_2 = \mathbf{u}_2 - \mathbf{v}_1 = -\mathbf{u}_1 + \mathbf{u}_2, \\ \mathbf{v}_3 &= \mathbf{u}_3 - \mathbf{v}_1 - 2\mathbf{v}_2 = \mathbf{u}_3 - \mathbf{u}_1 - 2(-\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3 \end{aligned} \quad (11)$$

i.e. (8) implies the inverse transformations

$$\mathbf{v}_1 = \mathbf{u}_1, \quad \mathbf{v}_2 = -\mathbf{u}_1 + \mathbf{u}_2, \quad \mathbf{v}_3 = \mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3. \quad (12)$$

Using (12) in (10) we get

$$\begin{aligned} \mathbf{v} &= a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = a\mathbf{u}_1 + b(-\mathbf{u}_1 + \mathbf{u}_2) + c(\mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3) \\ &= (a - b + c)\mathbf{u}_1 + (b - 2c)\mathbf{u}_2 + c\mathbf{u}_3. \end{aligned} \quad (13)$$

Thus, by definition,

$$(\mathbf{v})_{S'} = (a - b + c, b - 2c, c). \quad (14)$$

5) For the previous problem, what is the transition matrix $P_{SS'}$ **from** basis S' **to** basis S ? What is the transition matrix P_{SS} **from** basis S **to** basis S' ? Don't mix these two up. Maybe the easiest thing to do here is use the result of the last problem, but re-inventing can be useful as an independent check.

20pts

Solution

By definition, the transition matrix $P_{SS'}$ **from** basis S' **to** basis S behaves as

$$(\mathbf{v})_S = P_{SS'} (\mathbf{v})_{S'}, \quad (15)$$

where the coordinate vectors are obviously taken as column vectors. For problem 4), (15) is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = P_{SS'} \begin{bmatrix} a-b+c \\ b-2c \\ c \end{bmatrix}. \quad (16)$$

But, by relevant transformations, (16) gives,

$$\begin{aligned} P_{SS'} \begin{bmatrix} a-b+c \\ b-2c \\ c \end{bmatrix} &= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Leftrightarrow P_{SS'} \begin{bmatrix} a \\ b-2c \\ c \end{bmatrix} = \begin{bmatrix} a+b-c \\ b \\ c \end{bmatrix} \\ &\Leftrightarrow P_{SS'} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c \\ b+2c \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \end{aligned} \quad (17)$$

By making in turn choices of (a, b, c) corresponding to standard basis elements (of $\mathbb{R}^3$), on finds that we must have then that the columns of $P_{SS'}$ must be in turn the columns of the 3×3 matrix on the right of (17), i.e. $P_{SS'}$ must be that matrix: one finds

$$P_{SS'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}. \quad (18)$$

To find $P_{S'S}$, we note by definition it behaves as

$$(\mathbf{v})_{S'} = P_{S'S} (\mathbf{v})_S, \quad (19)$$

which for problem 1) is evidently

$$P_{S'S} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a-b+c \\ b-2c \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad (20)$$

and logic as above dictates

$$P_{S'S} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}. \quad (21)$$

One checks the matrices of (18) and (21) are inverses, and that

$$\begin{aligned} P_{SS'} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & P_{SS'} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \\ P_{SS'} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \end{aligned} \quad (22)$$

which patently agrees with (8). (And this indicates an alternate, perhaps better approach.)

6) The eigenvalues of

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & 2 \\ -1 & -3 & 5 \end{bmatrix} \quad (23)$$

are $\lambda = 1, 2$, and 3 . Compute the eigenspaces associated to each of these eigenvalues.
(Recall eigenspaces are subspaces, hence specified as the span of a basis.)

15pts

Solution

We have

$$\begin{aligned} E_1(A) &= \ker(1I - A) = \ker \begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & -2 \\ 1 & 3 & -4 \end{bmatrix} = \ker \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & -2 \\ 0 & 2 & -2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \end{aligned}$$

$$\begin{aligned} E_2(A) &= \ker(2I - A) = \ker \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & -2 \\ 1 & 3 & -3 \end{bmatrix} = \ker \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \end{aligned}$$

$$\begin{aligned} E_3(A) &= \ker(3I - A) = \ker \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -2 \\ 1 & 3 & -2 \end{bmatrix} = \ker \begin{bmatrix} 2 & 2 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}. \end{aligned}$$

- 7) Tell me whether the following statement is true or false (pick one), and tell me why it is true or false: "There exists a 4×5 matrix with 5 linearly independent columns."

5 points

Solution

Such a matrix would satisfy $\text{rank}(A) = \dim \text{Col}A = 5$. But this is impossible since $\text{rank}(A) = \dim \text{Row}A \leq$ the number of rows of A which is 4.

- 8) Tell me whether the following statement is true or false (pick one), and tell me why it is true or false: "There exists a 4×5 matrix with 4 linearly independent columns and whose null space is 2 dimensional."

5 points

Solution

Such a matrix would satisfy $\text{rank}(A) = \dim \text{Col}A \geq 4$ and have nullity 2. But this is impossible since, by theorem, $\text{rank}(A) + \text{nullity}(A) = \text{number of columns of } A = 5 < 4 + 2$.

- 9) Determine whether the Wronskian tells you if the following sets of vectors in $C^3(\mathbb{R})$ are linearly independent. If the Wronskian is inconclusive, say so.
a) $\{1, \sin x, \cos x\}$, b) $\{1, x-3, 2x^2+4\}$, c) $\{x, 3x+2, 9x-5\}$.

15 pts

Solution

We have

a)

$$\begin{aligned} W\{1, \sin x, \cos x\} &= \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix} = -\begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix} \\ &= -(\cos^2 x + \sin^2 x) = -1 \neq 0 \end{aligned}$$

for some (all) value(s) of x , whence the set is linearly independent,

b)

$$W\{1, x-3, 2x^2+4\} = \begin{vmatrix} 1 & x-3 & 2x^2+4 \\ 0 & 1 & 4x \\ 0 & 0 & 4 \end{vmatrix} = 4 \neq 0$$

for some (all) value(s) of x , whence the set is linearly independent, and

c)

$$W\{x, 3x+2, 9x-5\} = \begin{vmatrix} x & 3x+2 & 9x-5 \\ 1 & 3 & 9 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

for all values of x , whence use of the Wronskian is inconclusive (but it is clear that the set is linearly dependent directly from the definition since $33x - 5(3x+2) - 2(9x-5) = 0$ for every value of x).

- 10) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and let those vectors be orthogonal with respect to the Euclidean inner product. Prove the Pythagorean theorem $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$. Note in your calculation the step at which you use the vectors' orthogonality.

Solution

5pts

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + 2 \cdot 0 + \|\mathbf{v}\|^2 \quad (\text{Orthogonality}) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \end{aligned} \quad (0.24)$$

- 11) Three (and only three) rules fix the value of the determinant of a square matrix A :
a) The determinant of the relevant identity is chosen to be 1. b) The value of the determinant is negated by interchanging rows. c) The determinant “function” is linear in each of A 's rows **individually**. As in the homework, the latter means that for an $n \times n$ matrix A , we always get

$$\det \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_{k-1} \\ \mathbf{r}_k \\ \mathbf{r}_{k+1} \\ \vdots \\ \mathbf{r}_n \end{bmatrix} = \det \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_{k-1} \\ \alpha \mathbf{a} + \beta \mathbf{b} \\ \mathbf{r}_{k+1} \\ \vdots \\ \mathbf{r}_n \end{bmatrix} = \alpha \det \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_{k-1} \\ \mathbf{a} \\ \mathbf{r}_{k+1} \\ \vdots \\ \mathbf{r}_n \end{bmatrix} + \beta \det \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_{k-1} \\ \mathbf{b} \\ \mathbf{r}_{k+1} \\ \vdots \\ \mathbf{r}_n \end{bmatrix}. \quad (25)$$

Here it is meant that the k^{th} row vector $\mathbf{r}_k$, which is also the linear combination $\alpha \mathbf{a} + \beta \mathbf{b}$ of a row vector $\mathbf{a}$ and a row vector $\mathbf{b}$, could be from any row of the matrix, i.e. from any of the rows from 1 to n . Use these and only these rules to compute

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad (26)$$

Hint: Realize that rule b) implies that the determinant of a matrix with two rows identical is zero.

15 pts

Solution

By rule c), in particular linearity in the first row, we have

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} + b \det \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}. \quad (27)$$

By rule c) again, but now as linearity in second rows, then gives us

$$\begin{aligned} \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= a \det \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} + b \det \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \\ &= a \left(c \det \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) + b \left(c \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \det \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right). \end{aligned} \quad (28)$$

From rule b), including the implied hint, we then have

$$\begin{aligned} \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= a \left(c \cancel{\det \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} + d \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) + b \left(c \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \cancel{\det \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} \right) \\ &= a \left(c \cdot 0 + d \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) + b \left(c \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \cdot 0 \right) \\ &= ad \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + bc \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = ad \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - bc \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= (ad - bc) \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \quad (29)$$

Finally, from rule a) we get

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (ad - bc) \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (ad - bc) \cdot 1 = ad - bc. \quad (30)$$