

Math 313 Midterm I KEY
Spring 2011
sections 001 and 002
Instructor: Scott Glasgow

Write your name very clearly on this exam booklet. In this booklet, write your mathematics clearly, legibly, in big fonts, and, most important, “have a point”, i.e., make your work logically and even pedagogically acceptable. (Other human beings not already understanding 313 should be able to learn from your exam.) To avoid excessive erasing, first put your ideas together on scratch paper, then commit the logically acceptable fraction of your scratchings to this exam booklet. More is not necessarily better: say what you mean and mean what you say.

Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed. Also, I will not even speak of this exam to anyone else in *any* fashion, not even whether it was difficult or not. My signature below indicates I accept this obligation.

Signature:

(Exams without this signature will not be graded.)

- Put the following matrix in reduced row-echelon form via elementary row operations:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 4 & -2 & 1 & 4 \end{bmatrix}. \quad (1)$$

10pts

Solution

The row reduction may proceed as follows:

$$\begin{aligned} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 4 & -2 & 1 & 4 \end{bmatrix} \xrightarrow[R3-4R1]{R2-R1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & -6 & -3 & 0 \end{bmatrix} \xrightarrow[R3/(-3)]{R2/(-2)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{R3-2R2} \\ & \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{R1-R3} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{R1-R2} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}. \quad (2) \end{aligned}$$

However the row reduction proceeds, the row echelon form is unique—the last matrix indicated in (2) is ***the*** answer.

- Assuming A and B are invertible matrices of the same size, prove that

$$(AB)^{-1} = B^{-1}A^{-1}. \quad (3)$$

15 points

Solution

$B^{-1}A^{-1}$ is the inverse of AB if and only if

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I, \quad (4)$$

to which we first note that, by the associative property of matrix multiplication,

$$\begin{aligned}
(AB)(B^{-1}A^{-1}) &= A((BB^{-1})A^{-1}) \\
&\text{and} \\
(B^{-1}A^{-1})(AB) &= B^{-1}((A^{-1}A)B).
\end{aligned} \tag{5}$$

Then, by the definition of the inverses, in particular that an inverse is both a right and a left inverse, we have, respectively, that

$$\begin{aligned}
(AB)(B^{-1}A^{-1}) &= A((BB^{-1})A^{-1}) = A(IA^{-1}) \\
&\text{and} \\
(B^{-1}A^{-1})(AB) &= B^{-1}((A^{-1}A)B) = B^{-1}(IB).
\end{aligned} \tag{6}$$

Using now the fact that the identity matrix is in fact the “multiplicative identity” we get

$$\begin{aligned}
(AB)(B^{-1}A^{-1}) &= A(IA^{-1}) = AA^{-1} \\
&\text{and} \\
(B^{-1}A^{-1})(AB) &= B^{-1}(IB) = B^{-1}B.
\end{aligned} \tag{7}$$

Finally we use again the definition of the inverses. In particular, using that an inverse is both a right and a left inverse, we have, respectively, that

$$\begin{aligned}
(AB)(B^{-1}A^{-1}) &= AA^{-1} = I \\
&\text{and} \\
(B^{-1}A^{-1})(AB) &= B^{-1}B = I,
\end{aligned} \tag{8}$$

which is the required (4)

3. Use Gaussian elimination, noting along the way the various relevant row operations and their relationship to the (evolving calculation of) the determinant, to show that

$$\det \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} = (x-y)(y-z)(z-x). \tag{9}$$

Be sure to note along the way the various relevant row operations to your Gaussian elimination and their accurate relationship to the (evolving calculation of) the determinant.

15 points

Solution

We indicate the various Gaussian elimination row operations in the upper right-hand corner of the matrix (whose determinant is being calculated), making sure to use the correct relationship of such to the evolving determinant calculation:

$$\begin{aligned}
 \det \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} & \xrightarrow[R3-R1]{R2-R1} \det \begin{bmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{bmatrix} \xrightarrow[R3/(z-x)]{R2/(y-x)} \\
 &= (y-x)(z-x) \det \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{bmatrix} \xrightarrow{R3-R2} (y-x)(z-x) \det \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{bmatrix} \\
 &= (y-x)(z-x)(1 \cdot 1 \cdot (z-y)) = (x-y)(y-z)(z-x). \tag{10}
 \end{aligned}$$

In the final steps above we used that a) the determinant of a triangular matrix is the product of its diagonal entries, and b) some obvious algebraic properties of multiplication (of scalars).

4. By using the “permutation definition” of the determinant, prove that if square matrix B is the same as square matrix A , except that one of B ’s rows is a scalar multiple k of the corresponding row of A , then

$$\det B = k \det A. \tag{11}$$

15 points

Solution

Let it be the j^{th} row of the matrix B that is the same as the scalar k multiplied by the j^{th} row of the matrix A . Then, by definition of the determinant, and the relationship between the 2 rows (of matrix A and matrix B), we have

$$\begin{aligned}
 \det B &:= \sum_{p: [n] \rightarrow [n]} (-1)^{d(p)} b_{1p(1)} \cdots b_{jp(j)} \cdots b_{np(n)} = \sum_{p: [n] \rightarrow [n]} (-1)^{d(p)} a_{1p(1)} \cdots (ka_{jp(j)}) \cdots a_{np(n)} \\
 &= k \sum_{p: [n] \rightarrow [n]} (-1)^{d(p)} a_{1p(1)} \cdots a_{jp(j)} \cdots a_{np(n)} = k \det A. \tag{12}
 \end{aligned}$$

5. Prove that, if the matrix A is invertible, the system $A\mathbf{x} = \mathbf{b}$ has one and only one solution $\mathbf{x}$, namely $\mathbf{x} = A^{-1}\mathbf{b}$. (Warning: there are two things to prove here, namely a) that if the system has a solution, then it can only be $\mathbf{x} = A^{-1}\mathbf{b}$, and that b) $\mathbf{x} = A^{-1}\mathbf{b}$ actually does solve the system. Here then you will have addressed the “one and only one” issues in reverse order: you first show that a) there is at most one solution, and then b) that there is at least one solution, which, together, show that in fact there is exactly one solution. In parts a) and b) you will use that A^{-1} is A ’s left and right inverse, respectively.)

15 points

Solution

If the system has a solution $\mathbf{x}$, then, for any such $\mathbf{x}$, we may write

$$A\mathbf{x} = \mathbf{b} \quad (13)$$

(without implicitly lying), and then, by left application of A^{-1} to (13), as well as by the associative property of matrix multiplication, obtain that

$$A^{-1}\mathbf{b} = A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I\mathbf{x} = \mathbf{x}. \quad (14)$$

In (14) we also used that A^{-1} is a left inverse of A , as well as the fact that the so-called identity matrix I is in fact a “multiplicative identity”. Here then we have just showed that if (13) has a solution, it’s got to be $\mathbf{x} = A^{-1}\mathbf{b}$. Thus we have showed that (13) has at most one solution. But our demonstration does not yet preclude there being no solution. To do that, we confirm that, for the only promising candidate $\mathbf{x} = A^{-1}\mathbf{b}$, we get

$$A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}, \quad (15)$$

so that our candidate was successful. (Here we have used the associative property of matrix multiplication, the fact that A^{-1} is a right inverse of A , as well as the fact that the so-called identity matrix I is in fact a “multiplicative identity”.) Thus we have showed that the system has one and only one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

6. Prove that the product AB of two square matrices (of the same size) cannot be invertible if B is not invertible. That is, prove that

$$B \text{ is not invertible} \Rightarrow AB \text{ is not invertible.} \quad (16)$$

Hint: Consider the relationship between solutions $\mathbf{x}$ of the equation $B\mathbf{x} = \mathbf{0}$ and solutions $\mathbf{x}$ of the equation $(AB)\mathbf{x} = \mathbf{0}$, and use equivalent statements.

15 points

Solution

By equivalent statements and associativity of matrix multiplication we have

$$\begin{aligned} B \text{ is not invertible} &\stackrel{\text{eq. stat.s}}{\Leftrightarrow} \exists \mathbf{x} \neq \mathbf{0} \text{ s.t. } B\mathbf{x} = \mathbf{0} \\ &\Rightarrow \exists \mathbf{x} \neq \mathbf{0} \text{ s.t. } (AB)\mathbf{x} \stackrel{\text{assoc.}}{=} A(B\mathbf{x}) = A(\mathbf{0}) = \mathbf{0} \\ &\stackrel{\text{eq. stat.s}}{\Leftrightarrow} AB \text{ is not invertible.} \end{aligned} \tag{17}$$

7. Assume that both the matrix B and the matrix C are inverses of the matrix A . Show that B and C are just two aliases for the same matrix, i.e. show that in fact $B = C$.

15 points

Solution

The descriptions of B and C demand that

$$AB = BA = I = AC = CA. \tag{18}$$

Using the associative property of matrix multiplication in two different ways on the product BAC we get

$$\begin{aligned} BAC &= B(AC) = BI = B \\ &\text{and} \\ BAC &= (BA)C = IC = C, \end{aligned} \tag{19}$$

so that indeed

$$\begin{aligned} B &= BAC = C \\ &\Rightarrow \\ B &= C \end{aligned} \tag{20}$$

as claimed. Note that in (19) we also used that a) C is a right inverse of A , b) B is a left inverse of A , and that c) the identity matrix acts as both a right and left multiplicative identity.

8. Write the (invertible) matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad (21)$$

as a product of 2 elementary matrices.

10pts

Solution

Since

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad (22)$$

i.e., in what we hope is compelling notation, since

$$(R_1 - R_2)((R_2 - R_1)(A)) = I, \quad (23)$$

then clearly

$$\begin{aligned} A &= (R_2 - R_1)^{-1} \left((R_1 - R_2)^{-1} (I) \right) \stackrel{\text{theorem}}{=} (R_2 - R_1)^{-1} (I) (R_1 - R_2)^{-1} (I) \\ &= (R_2 + R_1) \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) (R_1 + R_2) \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \end{aligned} \quad (24)$$

Here the indicated theorem may itself not be clear—it requires a proof with 3 parts to it—but the rest should be clear. One partially checks this theorem by noting that in fact

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = A. \quad (25)$$

9. Find the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (26)$$

by row reducing $[A|I]$ to $[I|A^{-1}]$. Assume the parameters a, b, c , and d do not take on any special values, nor have a special relationship among them—that is row reduce naively, without worrying about any divisions by hidden zeros. Simplify your answer

completely. (Related: kA^{-1} should be exceptionally simple if scalar k is chosen to be $ad - bc$.)

15 points

Solution

The naïve row reduction mentioned proceeds as follows:

$$\begin{aligned}
 [A|I] &= \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right] \xrightarrow{aR2 - cR1} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad - bc & -c & a \end{array} \right] \xrightarrow{(ad - bc)R1 - bR2} \\
 &\sim \left[\begin{array}{cc|cc} a(ad - bc) & 0 & ad & -ab \\ 0 & ad - bc & -c & a \end{array} \right] \xrightarrow{R1/a} \left[\begin{array}{cc|cc} ad - bc & 0 & d & -b \\ 0 & ad - bc & -c & a \end{array} \right] \xrightarrow{R1/a} \\
 &\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ 0 & 1 & \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \end{array} \right] \xrightarrow{R1/a} [I|A^{-1}].
 \end{aligned} \tag{27}$$

10. When is the product of two symmetric matrices symmetric? Prove this. (Assume that the product of the two matrices makes sense, i.e. assume the two matrices are the same size.)

10 points

Solution

The product AB of two matrices A and B is, by definition, symmetric if and only if $AB = (AB)^T$. On the other hand we have proved that $(AB)^T = B^T A^T$, which, under the present circumstances, is the product BA . Thus, if the matrices A and B are symmetric, their product is symmetric if and only if $AB = BA$, i.e., if and only if A and B commute.

11. Suppose the system of equations

$$A\mathbf{x} = \mathbf{b} \tag{28}$$

has no solution $\mathbf{x} \in \mathbb{R}^n$ for some particular $\mathbf{b} \in \mathbb{R}^n$. (Here A is an $n \times n$ matrix.) Tell me whether the following statement is true or is false: “There is another right-hand-side $\mathbf{b} \in \mathbb{R}^n$ such that (28) has infinitely many solutions.” **Prove your assertion.**

15 points

Solution

By (the contrapositive/negation of our stated) equivalent statements the supposition gives us that, for example, $A\mathbf{x} = \mathbf{0}$ has more than just one solution. (Here we have chosen the “other $\mathbf{b}$ ” to be $\mathbf{0}$. Mind you $\mathbf{0}$ really is *another* $\mathbf{b}$ since for $\mathbf{b} = \mathbf{0}$ (28) actually has a solution, namely $\mathbf{x} = \mathbf{0}$, contrary to the supposition.) But since the possibilities for the number of solutions of linear systems are only 0, 1, or ∞ , there must be an infinite number of solutions for $\mathbf{b} = \mathbf{0}$, so that the statement is true.