Embedding into duo rings

Pace P. Nielsen

Abstract. We construct a domain that does not embed in a duo ring. (A ring is duo when its left and right ideals are all two-sided). On the other hand, we show that a large class of rings can be embedded in duo rings. This leads to the construction of a duo ring whose polynomial ring is not semicommutative (definition recalled below).

1. Basic information about duo rings

There are many generalizations of commutativity in ring theory. The purpose of this short note is to shed some light on the basic structural properties of the following natural, but not well-understood, classes of rings that are close to commutative.

Definition 1.1. A ring is *left duo* if every left ideal is a two-sided ideal. The *right duo* rings are defined dually, and a ring is *duo* if it is both left and right duo.

Every left ideal is a sum of principal left ideals, so the left duo property for R is equivalent to the elementwise statement

(1.2) $$\forall a, b \in R,\ \exists b' \in R,\ ab = b'a.$$

This fact was originally noted by Feller in the 1958 paper [5], in which the "duo" terminology was first introduced.

The left duo and right duo conditions are distinct; for a typical example see the first paragraph of [4]. (A more general class of examples is described by [10, Example 3.6]. The remark following that example is also important.) Duo rings are not always commutative; any division ring that is not a field is an example (such as the ring of real quaternions).

Another important generalization of commutativity is defined as follows.

Definition 1.3. A ring R is *semicommutative* when it satisfies the condition

$$\forall a, b \in R,\ ab = 0 \Longrightarrow aRb = 0.$$

Other terminology than "semicommutativity" is used on occasion for this property, especially in earlier papers. Readers are directed to the third paragraph on page 496 of [10] for a thorough listing of alternative terminology.

The next lemma is a well-known result that connects our two definitions.

Lemma 1.4. *Every left or right duo ring is semicommutative.*

Proof. It suffices to handle the case when R is left duo. Assume that $ab = 0$ for some $a, b \in R$. Given any $c \in R$, then there exists some $c' \in R$ with $ac = c'a$. Hence

$$acb = c'ab = 0,$$

thus demonstrating that R is semicommutative. □

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There exist semicommutative rings that are not left or right duo. Any simple domain that is not a division ring, such as the Weyl algebra over $\mathbb{R}$, demonstrates that the converse of Lemma 1.4 fails by an application of [9, Ex. 1.19]. (Other examples are described in [10].)

Summarizing, we have the following irreversible implication diagram for rings:

$$\textbf{(1.5)} \qquad \text{commutative} \Longrightarrow \text{duo} \Longrightarrow \text{left or right duo} \Longrightarrow \text{semicommutative.}$$

2. Non-embedding result

The semicommutativity property passes to subrings. Thus, if a ring R embeds into a duo ring, it must be the case that R is semicommutative. Is the converse possibly true? The answer turns out to be no, even for domains. The example we give is similar to an example of Mal'cev showing that cancellative semigroups do not generally embed in groups (see [8, Theorem 9.8]).

Example 2.1. *There exists a domain R that does not embed in a left or right duo ring.*

Construction. Let F be a field and let

$$R := F\langle a, b, c, d, e, f, g, h \,:\, cd = ab, f^2d = aeb, chd = ag^2\rangle.$$

Order the letters in the usual alphabetical order, and order monomial expressions in the shortlex order (i.e., first by total length, and then lexicographically). We may now think of the relations as the reduction rules

$$cd \mapsto ab, \qquad f^2d \mapsto aeb, \qquad chd \mapsto ag^2,$$

where monomial expressions are being replaced with earlier monomial expressions. The main conditions of the Diamond lemma [2] hold vacuously (there are no overlaps or inclusions to resolve), and so elements of R are put in a unique reduced form by repeatedly applying these reduction rules.

By the above description, R is not the zero ring. To finish showing that R is a domain, suppose by way of contradiction that $\alpha\beta = 0$ with $\alpha, \beta \in R \setminus \{0\}$ written in reduced form. Let α_0 and β_0 be the maximal monomials occurring in the supports of α and β respectively. To have $\alpha_0\beta_0$ cancel out in the product, the end of α_0 and the beginning of β_0 must be one of the five pairs $(c, d), (f, fd), (f^2, d), (c, hd), (ch, d)$. The cancelling term must arise when we multiply two words α_1 and β_1 (of the same degrees as α_0 and β_0 respectively) from the supports of α and β, where the end of α_1 and the beginning of β_1 together form one of the corresponding pairs $(a, b), (a, eb), (ae, b), (a, g^2), (ag, g)$. In each of these cases, $\alpha_1\beta_0$ cannot cancel in the product, yielding the needed contradiction. Therefore R is a domain.

If R were to embed in a left duo ring S, then there would exist some element $x \in S$ with $ae = xa$. In particular,

$$(f^2 - xc)d = f^2d - xcd = aeb - xab = aeb - aeb = 0.$$

Duo rings are semicommutative, hence $(f^2 - xc)hd$ should equal 0. However, we find

$$\textbf{(2.2)} \qquad (f^2 - xc)hd = f^2hd - xchd = f^2hd - xag^2 = f^2hd - aeg^2,$$

which is a non-zero element of R in reduced form. Thus, R cannot embed in a left duo ring. The ring R is isomorphic to its opposite ring, by the anti-isomorphism determined by $a \leftrightarrow d$, $b \leftrightarrow c$, $e \leftrightarrow h$, and $f \leftrightarrow g$, so R also cannot embed in a right duo ring. $\square$

3. Towards embedding in a duo ring

Recall that a ring R is *graded* (or, more fully, $\mathbb{N}$-graded) when there is given an internal direct sum decomposition of additive groups

$$R = \bigoplus_{i\in\mathbb{N}} R_i$$

such that $R_iR_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{N}$. (We note that a ring may have many different $\mathbb{N}$-gradings.) We call R_i the *grade i component* of the graded ring. Elements of R_i are also said to *have grade i*, or be *homogeneous of degree i* (if there is no confusion with other notions of degree, such as in a polynomial ring). If S is another graded ring, with decomposition $S = \bigoplus_{i\in\mathbb{N}} S_i$, then we say that R is a *graded subring* of S if R is a unital subring of S and if $R_i \subseteq S_i$ for each $i \in \mathbb{N}$.

The ring R constructed in Example 2.1 is a graded F-algebra, by treating the multiplicative identity of R as having grade 0 and treating the letters generating R as having grade 1. As we discovered in the construction of R, an obstruction to embedding in a duo ring arose from a certain nonzero element in grade 4 that appeared in (2.2). We will prove the surprising fact that all such obstructions disappear when large grades are trivial.

The following lemma collects some basic information about the types of graded rings we will work with.

Lemma 3.1. *Let F be a field, and let R be a graded ring such that*

- *the grade 0 component is contained in the center of R and equals F as a ring, and*
- *the grade 4 and higher components are all zero.*

In this situation, each element of R is either nilpotent or a unit, depending only on whether its grade 0 component is zero or not. Moreover, R is semicommutative if it merely satisfies the semicommutativity property for grade 1 elements, meaning that $rs = 0$ implies $rts = 0$ whenever $r, s, t \in R_1$.

Proof. Let S be the set of elements whose grade 0 component is zero. By the second bullet point S is a nilpotent set, with nilpotence index at most 4. As S is an ideal, it is contained in the Jacobson radical. Any $u \in F \setminus \{0\}$ is a unit in R, and so all elements of the form $u + s$ with $s \in S$ are also units. Applying the first bullet point, this finishes the proof that each element of R is either nilpotent or a unit.

For the last sentence of the lemma, assume grade 1 elements satisfy the semicommutativity property, and let $a, b, c \in R$ with $ab = 0$. Our goal is to show $acb = 0$.

Using the second bullet point, write $a = a_0 + a_1 + a_2 + a_3$, with $a_i \in R_i$. Similarly write decompositions $b = b_0 + b_1 + b_2 + b_3$ and $c = c_0 + c_1 + c_2 + c_3$, with $b_i, c_i \in R_i$. If $a_0 \neq 0$, then a is a unit, and from $ab = 0$ we get $b = 0$. In that case, the equality $acb = 0$ is trivial. Thus, we reduce to the case $a_0 = 0$, and by a dual argument we also reduce to $b_0 = 0$.

From $ab = 0$ we have $a_1b_1 = 0$. Therefore, we find

$$acb = ac_0b + a_1c_1b_1 + (\text{other terms of grade at least } 4) = 0,$$

since c_0 is central and R is semicommutative in grade 1. □

Ultimately, we will describe a class of graded rings that embed in duo rings. It is much easier to first handle the simpler case of left duo rings. In fact, we will simplify the situation even further, by first handling a very weak form of (1.2) where only a fixed pair of elements $a, b \in R$ are allowed. This important first step is stated as follows.

Proposition 3.2. *Let F be a field. If R is a graded ring such that*

- *the grade 0 component is contained in the center and equals F as a ring,*
- *the grade 4 and higher components are all zero, and*
- *the ring is semicommutative,*

then for each pair of elements $a, b \in R$, there exists a graded ring S (depending on the pair) such that R is a graded subring of S, the condition

$$\exists b' \in S, \; ab = b'a \tag{3.3}$$

holds, and S satisfies all three bulleted conditions. Moreover, the rings R and S share the same nontrivial grade 1 zero-products, meaning that if $s, s' \in S \setminus \{0\}$ are grade 1 elements satisfying $ss' = 0$, then $s, s' \in R$.

Proof. Our first goal is to think about R in a slightly simpler form. Fix an F-basis $\mathscr{B}$ for R_1, and let $\mathscr{B}'$ be a homogeneous F-basis for $R_2 \oplus R_3$. Consider the free (unital) F-algebra

$$P := F\langle x_\alpha \, : \, \alpha \in \mathscr{B} \cup \mathscr{B}' \rangle.$$

The map $x_\alpha \mapsto \alpha$ extends to an F-algebra homomorphism $P \to R$. This homomorphism is surjective since

$$\{1\} \cup \mathscr{B} \cup \mathscr{B}'$$

generates R as an F-algebra. Letting I be the kernel, then by the first isomorphism theorem $P/I \cong R$. We will write cosets in P/I using bar notation or coset notation, depending on which is more convenient at the given moment.

Treat P as a graded ring by taking x_α to have the same grade as α, for each $\alpha \in \mathscr{B} \cup \mathscr{B}'$. With this grading fixed, the ideal I is homogeneous, and we may as well identify P/I with R (after identifying F with the grade 0 component of P/I). Of particular importance is to note that there are no nonzero elements of grade 1 in I, since $\mathscr{B}$ was originally chosen as a basis for R_1. (Of course, after replacing R by P/I, we then treat $\mathscr{B} \cup \mathscr{B}'$ merely as an indexing set for the given graded generators of P.)

Fix $a, b \in R$. Write $a = \overline{p}$ and $b = \overline{q}$, for some $p, q \in P$. Further, we may assume that $p = p_0 + p_1 + p_2 + p_3$ with each $p_i \in P_i$; the higher grades may be taken to equal zero since we only need to choose a representative of the coset a. If there exists some element $b' \in R$ such that the equality $ab = b'a$ already holds, then we can just take $S := R$. For the remainder of the proof we will assume:

$$\boxed{\text{There is no element } b' \in R \text{ such that } ab = b'a.} \tag{3.4}$$

In particular $a \neq 0$, else we could take $b' := 0$. Also, a cannot be a unit in R, because otherwise $ab = (aba^{-1})a$ and so we could have taken $b' := aba^{-1} \in R$. Consequently, by Lemma 3.1, we have $p_0 = 0$.

Our next simplification arises by noting that q, the given representative of the coset b, is a finite sum of monomials in finitely many of the generators for P. Think of (3.3) as saying that a can be pushed past b, at the cost of modifying b to b'. If we can push a past $\overline{x_\alpha}$ for each indeterminate x_α appearing in q, then we are done. Thus, up to repeating the argument finitely many times, it suffices to deal with the case when:

$$\boxed{q = x_\alpha \text{ for some } \alpha \in \mathscr{B} \cup \mathscr{B}'.} \tag{3.5}$$

Suppose, for a moment, that we can find a ring S with all the stated conditions of this proposition, in the special case when $p_3 = 0$. As $b = \overline{q}$ has a trivial grade 0 component, the same must be true of b', and consequently $bR_3 = 0 = R_3b'$. Thus, this choice of b' continues to satisfy (3.3) if a is replaced by any element in $a+R_3$. Therefore, without loss of generality, we may assume $p_3 = 0$.

We now break into cases, with the first case being the hardest.

Case 1: Assume that $p_2 = 0$ and that q has grade 1. Thus, $p = p_1 \in R_1$.

Let P' be the free F-algebra generated by the same generators as P together with a single new indeterminate v. Giving v grade 1, and giving the other variables their previous grades, this makes P a graded subring of P'.

Let J be the ideal of P' generated by

(1) the set I,
(2) the element $pq - vp$,
(3) the elements rvr' whenever $r, r' \in P_1$ satisfy $rr' \in I$, and
(4) the homogeneous elements of grade 4 or more.

The ideal J is homogeneous, and so $S := P'/J$ is a graded ring. Since $I \subseteq J$, the inclusion map $P \to P'$ induces a ring homomorphism $R \to S$ given by the rule $x + I \mapsto x + J$, for each $x \in P$. Our next goal is to verify that this map is injective.

Fix $x \in P$, and suppose $x \in J$. Our goal is to show that $x \in I$. Since J is homogeneous, we may as well assume that x is homogeneous, and further that it has grade smaller than 4.

There are no elements of J of grades 0 or 1 (as there are none in I). If x has grade 2, it is of the form

$$x = z + c(pq - vp)$$

for some $z \in I \cap P_2$ and some $c \in F$. Since $x \in P$, no monomial involving v occurs in x when fully simplified, and so $c = 0$, and thus $x = z \in I$.

Finally, if x has grade 3, it is of the form

$$x = z + vz' + z''v + s(pq - vp) + (pq - vp)t + cv(pq - vp) + c'(pq - vp)v + \sum_{k=1}^{m} r_k v r'_k$$

for some $z \in I \cap P_3$, some $z', z'' \in I \cap P_2$, some $s, t \in P_1$, some $c, c' \in F$, and some $r_1, \ldots, r_m, r'_1, \ldots r'_m \in P_1$ with $r_i r'_i \in I$ for each positive integer $i \leq m$. Once again, since $x \in P$, no monomial involving v occurs in x when fully simplified. By considering monomials having two or more v's in them, $c = c' = 0$. Next, considering monomials ending in v, we see that $z'' = 0$. After these reductions, considering monomials of degree 3 starting with v forces $pt = z' \in I$, and by semicommutativity of R then $pqt \in I$. (Important note: We know that only semicommutative rings can possibly embed in duo rings. So semicommutativity of R had to play an essential role somewhere in our computations. This is the place where we are using it.) With both $pqt \in I$ and $vz' - vpt = 0$, we may remove the $(pq - vp)t$ and vz' terms from x by incorporating pqt into z.

The remaining terms in x involving v are $-svp + \sum_{k=1}^{m} r_k v r'_k$, so this sum is zero in P'. By specializing to $v = 1$, we thus have $sp = \sum_{k=1}^{m} r_k r'_k \in I$. Consequently, $spq \in I$. After absorbing this term into z, as well as removing all monomials involving v from x (since $x \in P$), we have $x = z \in I$, as desired.

The ring $S = P'/J$ thus contains a natural copy of $R = P/I$ as a graded subring. (Standard set-theoretic machinery allows us to use this embedding to replace S with an isomorphic copy of itself that literally contains R as a graded subring.)

The first bullet point of this proposition clearly holds for S. By condition (4) defining J, the second bullet point holds for S. Next, (3.3) holds for S by condition (2). To prove that S is semicommutative, it suffices (by Lemma 3.1, together with the semicommutativity of R and condition (3) defining J) to verify the final sentence in the proposition.

Suppose that $s, s' \in S_1 \setminus \{0\}$ satisfy $ss' = 0$. Writing $s = x + J$ and $s' = x' + J$, for some $x, x' \in P_1' \setminus \{0\}$, this means that $xx' \in J$. Write $x = y + cv$ and $x' = y' + c'v$, for some $y, y' \in P_1$ and some $c, c' \in F$. If $c = c' = 0$ we are done. If $c' \neq 0$, then since $x \neq 0$, we find that xx' contains a grade 2 monomial ending with v, but there is no such element in J, yielding a contradiction. We are thus reduced to handling the case when $c' = 0$ and $c = 1$ (after scaling x). Since $s' \neq 0$, we have $y' \neq 0$. We need $yy' + vy' \in J$, and the only way this can happen (by looking at the generators for J) is if $y' \in Fp \setminus \{0\}$. We may assume $y' = p$ (after scaling x'). This means that

$$yp + pq = yp + vp + (-vp + pq) = xx' + (-vp + pq) \in J \cap P = I$$

contradicting (3.4). This finishes Case 1.

Case 2: Assume that q has grade at least 2.

If q has grade 3, then since a is not a unit, we have $ab = 0$. This contradicts (3.4), since we could have taken $b' := 0$.

Thus, q has grade 2. Similar arguments further reduce us to the case when $p = p_1 \in R_1$. We now repeat the construction as in Case 1, except that v now has grade 2, and the definition of J simplifies by removing condition (3), since it is subsumed by condition (4). The remaining details are left to the reader (and they are almost trivial in comparison to the work needed in Case 1).

Case 3: Assume that q has grade 1 and that $p_1 = 0$. Thus, $p = p_2 \in R_2$.

This case is also similar to Case 1, with v having grade 1 but p having grade 2.

Case 4: Assume q has grade 1 and that $p_1, p_2 \neq 0$.

This case is somewhat different than the others. First, by applying Case 1 if necessary (and thus possibly enlarging R), we may as well assume that there is some $z \in P_1$ such that $\overline{p_1}b = \overline{zp_1}$. Now, we repeat the same construction of S as in the previous cases, with v once again having grade 2 (so that condition (3) is irrelevant), except that condition (2) is changed to asserting that the element $vp_1 + zp_2 - p_2q$ is in J. This will guarantee that by taking $b' = \overline{z + v}$ we fulfill (3.3). (Second important note: Even though b is homogeneous of grade 1, the elements a and b' are not. The biggest conceptual hurdle in proving this proposition was realizing that b' did not need to be homogeneous, even though b is!) $\square$

4. The full embedding theorem

Proposition 3.2 lets us pass from a ring R to another ring S, without losing any relevant properties, and the ring S is *slightly* closer to being left duo. One might say that a pair of elements $a, b \in R$ has been "left duoized" in S. We can now easily left duoize all pairs in R.

Corollary 4.1. *The statement of* Proposition 3.2 *holds if the phrase "for each pair of elements $a, b \in R$" is moved to the beginning of* (3.3) *and if the parenthetical phrase "(depending on the pair)" is removed.*

Proof. Let R_0 be a ring satisfying the bulleted conditions of Proposition 3.2. Let α be an ordinal such that (a_i, b_i) ranges over all pairs in $R_0 \times R_0$ as i ranges over elements in α.

We construct a chain of rings S_i, each satisfying the three bullet points in Proposition 3.2, and each having the same grade 1 zero-products as R_0, as follows. For each $i \in \alpha$, the ring S_i will be the ring S guaranteed by Proposition 3.2 for the pair (a_i, b_i) over a ring R_i. We already defined R_i when $i = 0$. When i is the successor of j, take $R_i := S_j$. Finally, when i is an infinite limit, take $R_i := \bigcup_{j<i} S_j$. (Note that the conditions of Proposition 3.2 on the ring S all pass through unions.) The ring $T := \bigcup_{i\in\alpha} S_i$ will satisfy

$$\forall a, b \in R,\ \exists b' \in T,\ ab = b'a$$

as well as all the other needed conditions. □

We are now ready for the full embedding theorem.

Theorem 4.2. *Let F be a field. If R is a graded ring such that*

- *the grade 0 component is contained in the center and equals F as a ring,*
- *the grade 4 and higher components are all zero, and*
- *the ring is semicommutative,*

then R embeds in a duo ring.

Proof. First, let S_0 be the ring guaranteed by Corollary 4.1 in which all pairs of elements from R have been left duoized (and no new nontrivial grade 1 zero-products occur, and the three bullet points still hold). The hypotheses of Proposition 3.2 and Corollary 4.1 are left-right symmetric. So by the dual of Corollary 4.1, we can let T_0 be a ring extension of S_0, where all the pairs in S_0 (and hence in R) have been right duoized.

Repeating this process countably many times results in a chain of graded ring extensions

$$T_0 \subseteq T_1 \subseteq \dots.$$

Taking $T := \bigcup_{i\in\mathbb{N}} T_i$, we claim that this ring is duo. Indeed, for each pair of elements $a, b \in T$, there is some index $i \in \mathbb{N}$ such that $a, b \in T_i$. This pair has been both left and right duoized in $T_{i+1} \subseteq T$. □

5. A new example

Over polynomial rings (and even skew polynomial rings), there is a significant collapse among the conditions in (1.5). As a special consequence of [11, Theorem 1], it happens that $R[x]$ is left or right duo if and only if R is commutative. However, $R[x]$ can be semicommutative without being commutative; for instance, if R is domain that is not commutative, then $R[x]$ is a domain and hence semicommutative, but it is not commutative.

Semicommutativity does not pass to polynomial rings, unlike commutativity. The first example appeared as [6, Example 2], and is constructed as follows. Taking the free unital $\mathbb{F}_2$-algebra

$$\mathbb{F}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle,$$

first factor by the ideal forcing $f(x)g(x) = 0$, for the polynomials $f(x) := a_0 + a_1x + a_2x^2$ and $g(x) := b_0 + b_1x + b_2x^2$. Finally, factor by the relations forcing semicommutativity and forcing all monomials of degree 4 to be zero. One checks, using the Diamond lemma, that $f(x)cg(x) \neq 0$ in this final factor ring.

This work leaves open the question: Is $R[x]$ semicommutative when R is duo? This was raised in 2008 as [3, Question 11.2], and later as [1, Question 4.1]. We can now answer this question in the negative.

The semicommutative ring R constructed in [6, Example 2] satisfies the three bullet points of Theorem 4.2. (Both [7, Example 2.1] and [3, Example 8.6] also satisfy the bullet points. We leave it to the reader to decide which ring is easiest to work with.) Thus, R embeds in a duo ring S. However, since $R[x]$ is not semicommutative, neither is $S[x]$.

6. Final thoughts

There are potential ways in which the example from Section 5 might be improved. For instance, one might ask if there exists a *finite* duo ring whose polynomial ring is not semicommutative.

Another question concerns the *symmetric* property. Let R be a symmetric ring, which means that $abc = 0 \Longrightarrow acb = 0$, for all $a, b, c \in R$. This is a strictly stronger condition than semicommutativity. Can we improve Proposition 3.2 so that S is also symmetric when R is symmetric? One may need to add new relations in grade 3 in the construction of S, and it is unclear whether or not this affects the embeddability of R into S.

Other questions remain. For example, it would also be interesting to characterize other obstructions to embedding in duo rings, besides just the one given by (2.2).

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Department of Mathematics, Brigham Young University, Provo, UT 84602, USA
Email address: pace@math.byu.edu