

LINEAR ALGEBRA BOOT CAMP

WEEK 4: THE SPECTRAL THEOREM

Unless otherwise stated, all vector spaces in this worksheet are finite dimensional and the scalar field $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 1. A linear operator $L : V \rightarrow V$ on a finite dimensional vector space is called self-adjoint if $L^* = L$ and skew-adjoint if $L^* = -L$. When $\mathbb{F} = \mathbb{R}$ we use the terms symmetric and skew-symmetric, and when $\mathbb{F} = \mathbb{C}$ we use the terms Hermitian and skew-Hermitian.

Definition 2. A positive definite Hermitian inner product is just what we have been calling an inner product all along.

Definition 3. A transformation L is a positive definite Hermitian transformation if it is Hermitian (or symmetric in the real case) and satisfies $(Lx, x) > 0$ for $x \in V \setminus \{0\}$.

Definition 4. A linear map $L : V \rightarrow V$ is said to be completely reducible if every invariant subspace has a complementary invariant subspace.

Theorem 5. (If you didn't prove this last week, you should prove it now)

The following are equivalent:

- (1) $\|L(x)\| = \|x\|$ for all $x \in V$.
- (2) $(L(x), L(y)) = (x, y)$ for all $x, y \in V$.
- (3) $L^*L = 1$
- (4) $LL^* = 1$
- (5) L takes orthonormal sets to orthonormal sets.

Corollary 6. L is an isometry (an isomorphism satisfying (2) above) if and only if $L^* = L^{-1}$.

Definition 7. If $V = W = \mathbb{R}^n$, the isometries are called orthogonal matrices. If $V = W = \mathbb{C}^n$, the isometries are called unitary matrices. (The sets O_n of orthogonal matrices and U_n of unitary matrices are groups, but we won't use that here).

Definition 8. A linear operator $L : V \rightarrow V$ is normal if $LL^* = L^*L$.

Warmup. I should have put this on last week's worksheet.

- (1) Consider a 3×3 real symmetric matrix with determinant 6. Assume $(1, 2, 3)$ and $(0, 3, -2)$ are eigenvectors with eigenvalues 1 and 2, respectively.
 - (a) Give (with proof) an eigenvector of the form $(1, x, y)$ for some real x, y which is linearly independent of the two vectors above.

(b) What is the eigenvalue of this eigenvector?

F03-9

Preliminaries.

- (2) (Self-and Skew-Adjoint Operators are Completely Reducible) Prove that if $L : V \rightarrow V$ is self- or skew-adjoint then for each invariant subspace $M \subset V$ the orthogonal complement $M^\perp$ is also invariant.
- (3) (Isometries are Completely Reducible) Prove that if $L : V \rightarrow V$ is an isometry then for each invariant subspace $M \subset V$ the orthogonal complement $M^\perp$ is also invariant.
- (4) Suppose that L is self-adjoint. Prove that $L = 0$ if and only if $(L(x), x) = 0$ for all $x \in V$.
Hint: look at $(L(x + y), x + y)$.
- (5) Suppose that $\mathbb{F} = \mathbb{C}$. Prove that $L = 0$ if and only if $(L(x), x) = 0$ for all $x \in V$. *Hint:* use the previous hint with $x + y$ and $x + iy$.
- (6) Suppose that T is a normal operator. Prove that if v is an eigenvector for T then v is also an eigenvector for T^* .
S07-5
- (7) Prove the following characterization of normal operators $L : V \rightarrow V$.
 The following are equivalent:
- (a) $LL^* = L^*L$.
 - (b) $\|L(x)\| = \|L^*(x)\|$ for all $x \in V$.
 - (c) $AB = BA$, where $A = \frac{1}{2}(L + L^*)$ and $B = \frac{1}{2}(L - L^*)$.

The Spectral Theorem. In this section we will prove various versions of The Spectral Theorem in several different ways. You already saw a proof of The Spectral Theorem over $\mathbb{R}$ last week (#37), but we will see a few more here. In each case, we will use the same basic outline:

- find an eigenvector $x \neq 0$ for L^* ,
- show that $(\text{span } x)^\perp$ is L -invariant,
- lather, rinse, and repeat.

(8) Prove the Spectral Theorem over $\mathbb{C}$:

Theorem 9. Let $L : V \rightarrow V$ be a self-adjoint (i.e. Hermitian) linear operator on a complex inner product space. Then there exists an orthonormal basis of eigenvectors with real eigenvalues.

S11-3

(9) Show that if L is a complex skew-adjoint operator, then iL is self-adjoint. Conclude that the Spectral Theorem also holds for complex skew-adjoint operators. In this case, what do the eigenvalues look like?

(10) Suppose that $T : V \rightarrow V$ is Hermitian and that the matrix representation of T^2 in the standard basis has trace zero. Prove that $T = 0$.

S03-10

(11) Prove that commuting Hermitian operators are simultaneously orthogonally diagonalizable (i.e. there is an orthonormal basis consisting of eigenvectors for both operators).

S02-11, S10-3

(12) Let Y be an arbitrary set of commuting matrices in $M_n(\mathbb{C})$. Prove that there exists a non-zero vector v which is a common eigenvector for all elements of Y .

S06-10

(13) Let A be a Hermitian $n \times n$ complex matrix. Show that if $(Av, v) \geq 0$ for all $v \in \mathbb{C}^n$ then there exists an $n \times n$ matrix T so that $A = T^*T$.

S05-3

- (14) In this exercise we'll prove the Spectral Theorem over $\mathbb{R}$ several times. F01-9, S04-10, S06-9

Theorem 10. Let $L : V \rightarrow V$ be a symmetric linear operator on a real inner product space. Then there exists an orthonormal basis of eigenvectors with real eigenvalues.

Note that the proof is identical to the proof in the complex case once we know that we can always find an eigenvector. So we'll actually prove the following:

Theorem 11. Let $L : V \rightarrow V$ be a symmetric linear operator on a real inner product space. Then L has an eigenvector with a real eigenvalue.

There are several reasonable ways to do this, and this seems like a good place to list a few.

- (a) Let $S = \{x \in V : (x, x) = 1\}$. Choose $x \in S$ such that $(Lx, x) = \sup_S (Ly, y)$. Prove that if $(x, y) = 0$ then $(Lx, y) = 0$. Conclude that x is an eigenvector for L with eigenvalue (Lx, x) . (You did this last week, so skip it if you want.)

- (b) With S, x as in (a), the real-valued function

$$f(y) = \frac{(Ly, y)}{\|y\|^2}$$

has a maximum at $y = x$ and the maximum value is $\lambda = (Lx, x)$. So for any $y \in V$, the function $\phi(t) = f(x + ty)$ has a maximum at $t = 0$. Compute the derivative $\phi'(t)$ and use its value at $t = 0$ to show that x is an eigenvector with eigenvalue λ .

- (c) Prove that $L^2 + bL + c$ is invertible when L is symmetric and $b^2 < 4c$. (*Hint:* Show that $((L^2 + bL + c)v, v) > 0$ for all $0 \neq v \in V$. Use this to conclude that L has an eigenvector with a real eigenvalue.)
- (d) Choose an orthonormal basis for V and let $A = [L] \in M_n(\mathbb{R})$ in this basis. Thinking of A as an element of $M_n(\mathbb{C})$, show that any eigenvalue of A must be real. Conclude that L has an eigenvector with a real eigenvalue.

- (15) Let $L : V \rightarrow V$ be a self-adjoint linear operator. Let $\mu \in \mathbb{C}$ and $\varepsilon > 0$ be given. Suppose that there is a unit vector $x \in V$ such that

$$\|L(x) - \mu x\| \leq \varepsilon.$$

Prove that L has an eigenvalue λ with $|\lambda - \mu| \leq \varepsilon$.

F11-9

- (16) Suppose that A is a symmetric $n \times n$ real matrix with eigenvalues $\lambda_1, \dots, \lambda_k$ (here $k \leq n$ and $\lambda_j \neq \lambda_k$ if $j \neq k$). Find the sets

$$X = \left\{ x \in \mathbb{R}^n : \lim_{k \rightarrow \infty} (x^t A^{2k} x)^{1/k} \text{ exists} \right\}$$

$$L = \left\{ \lim_{k \rightarrow \infty} (x^t A^{2k} x)^{1/k} : x \in X \right\}.$$

S07-4

- (17) (a) Let $A = (a_{ij})$ be an $n \times n$ real symmetric matrix with $\sum_{i,j} a_{ij} x_i x_j \leq 0$ for every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Prove that if $\text{tr}(A) = 0$ then $A = 0$.
 (b) Let T be a linear transformation in a complex vector space V with a positive definite Hermitian inner product. Suppose that $TT^* = 4T - 3I$. Prove that T is positive definite Hermitian and find all possible eigenvalues of T .

S10-4

- (18) For each of the following three fields (separately), is it true that every symmetric matrix $A \in M_2(\mathbb{F})$ is diagonalizable?
 (a) $\mathbb{F} = \mathbb{R}$
 (b) $\mathbb{F} = \mathbb{C}$
 (c) $\mathbb{F} = \mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$, the field with three elements.

Supply proofs/counterexamples (or cite the relevant theorems) for all parts.

F17-6

- (19) Prove the Spectral Theorem for Normal Operators. F02-10

Theorem 12. Let V be a complex inner product space. Then the linear operator $L : V \rightarrow V$ is normal if and only if there is an orthonormal basis of eigenvectors.

For the direction $\implies$: instead of invoking the Fundamental Theorem of Algebra to find an eigenvector, do the following. (This sort of reasoning will be useful later.)

- (a) Decompose $L = A + iB = \frac{1}{2}(L + L^*) + i\frac{1}{2i}(L - L^*)$. Find an eigenvalue α for A and, within the α -eigenspace for A , find an eigenvector x for B with eigenvalue β . Conclude that $\alpha + i\beta$ is an eigenvalue for L .
 (b) Show that $\text{span}\{x\}$ is L - and L^* -invariant, and show that $M = (\text{span}\{x\})^\perp$ is also L - and L^* -invariant. Conclude that $(L|_M)^* = L^*|_M$ and finish the proof.

- (20) Show that if $A \in M_n(\mathbb{C})$ is normal then $A^* = P(A)$ for some polynomial $P(x) \in \mathbb{C}[x]$.

S14-6

- (21) Mimic the proof of the Spectral Theorem for Normal Operators and prove the following.
Hint: if $A = \frac{1}{2}(L + L^T)$ and $B = \frac{1}{2}(L - L^T)$ then $L = A + B$, where A and B commute, A is symmetric, and B is skew-symmetric. Use that B^2 is symmetric.

Theorem 13. Let $L : V \rightarrow V$ be a normal operator on a real inner product space. Then there is an orthonormal basis

$$e_1, \dots, e_k, x_1, y_1, \dots, x_\ell, y_\ell,$$

where $k + 2\ell = n$ and

$$\begin{aligned} L(e_i) &= \lambda_i e_i, \\ L(x_j) &= \alpha_j x_j + \beta_j y_j, \\ L(y_j) &= -\beta_j x_j + \alpha_j y_j, \end{aligned}$$

where $\lambda_i, \alpha_j, \beta_j \in \mathbb{R}$.

- (22) Even when the operator $L : V \rightarrow V$ has no special properties which ensure an orthonormal basis of eigenvectors, we can still get a reasonably nice basis such that $[L]$ is upper triangular. Prove Schur's Theorem:

Theorem 14 (Schur's Theorem). Let $L : V \rightarrow V$ be an operator on a complex inner product space. Then there is an orthonormal basis $e_1, \dots, e_n$ such that

$$L = [e_1 \cdots e_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} [e_1 \cdots e_n]^*.$$

W02-11, W06-10

Quadratic Forms. A quadratic form is a function $Q : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$Q(x) = (Ax, x),$$

where A is a symmetric, real matrix. We say that Q is positive (resp. negative) definite if $Q(x) > 0$ (resp. $Q(x) < 0$) for all $x \neq 0$. If $>$ is replaced by $\geq$ (or $<$ by $\leq$) then we say Q is positive (negative) semidefinite. A symmetric matrix is positive/negative definite/semidefinite if the corresponding quadratic form has that property.

- (23) Let M be an $n \times n$ real matrix. Suppose that M is orthogonal and symmetric.
 (a) Prove that if M is positive definite then M is the identity.
 (b) Does the answer change if M is only positive semidefinite?

S17-3

(24) Let A, B be two positive definite 2×2 matrices. Prove or disprove:

- (a) $A + B$ is also positive definite.
- (b) $AB + BA$ is also positive definite.

S18-5

(25) Let $Q(x)$ be a quadratic form on $\mathbb{R}^n$. Show that there is an orthogonal basis where

$$Q(z) = -z_1^2 - \dots - z_k^2 + z_{k+1}^2 + \dots + z_\ell^2,$$

where $0 \leq k \leq \ell \leq n$.

(26) Prove the following special case of Descartes' Rule of Signs:

Theorem 15. Let

$$p(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0 = (t - \lambda_1) \cdots (t - \lambda_n),$$

where $a_i, \lambda_i \in \mathbb{R}$.

- (a) All roots of p are negative if and only if $a_i > 0$ for all i .
- (b) All roots of p are positive if and only if $a_{n-1} < 0$ and $\text{sign}(a_i)$ alternates.

(27) Let

$$f(x, y, z) = 9x^2 + 6y^2 + 6z^2 + 12xy - 10xz - 2yz.$$

Is there a point (x, y, z) such that $f(x, y, z) < 0$?

S15-7

(28) A bilinear form on a vector space V is a function $B : V \times V \rightarrow \mathbb{F}$ such that $x \mapsto B(x, y)$ and $y \mapsto B(x, y)$ are both linear. Show that the quadratic form Q always looks like $Q(x) = B(x, x)$ where B is a symmetric bilinear form (i.e. $B(x, y) = B(y, x)$).

Orthogonal Transformations.

(29) Suppose A is a real $n \times n$ orthogonal matrix.

- (a) Show that A is similar to a matrix which consists of 2×2 blocks down the diagonal along with some diagonal elements that are ± 1 with the 2×2 blocks being rotation matrices of the form

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

- (b) Use part (a) to show that if n is odd, then there is a nonzero vector v such that $A^2v = v$.

S13-9

- (30) An orthogonal $n \times n$ matrix A is called elementary if the corresponding linear transformation $L_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ fixes an $(n - 2)$ -dimensional subspace. Prove that every orthogonal matrix M is a product of at most $(n - 1)$ elementary orthogonal matrices.

S16-7

- (31) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the rotation by 60° counterclockwise about the plane perpendicular to the vector $(1, 1, 1)$ and let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the reflection about the plane perpendicular to the vector $(1, 0, 1)$. Determine the matrix representation of $S \circ T$ in the standard basis. You do not have to multiply the resulting matrices but you must determine any inverses that arise.

F01-8

- (32) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the rotation by 60° counterclockwise about the plane perpendicular to the vector $(1, 1, 2)$.

(a) Determine the matrix representation of T in the standard basis. Find all eigenvalues and eigenspaces of T .

(b) What are the eigenvalues and eigenspaces of T if $\mathbb{R}^3$ is replaced by $\mathbb{C}^3$?

F01-8, ~S04-8

- (33) Find the matrix representation in the standard basis for either rotation by an angle θ in the plane perpendicular to the subspace spanned by the vectors $(1, 1, 1, 1)$ and $(1, 1, 1, 0)$ in $\mathbb{R}^4$. You do not have to multiply the resulting matrices but you must determine any inverses that arise.

S02-9

Extra Problems.

- (34) Let H be an $n \times n$ Hermitian matrix with non-zero determinant. Use H to define an Hermitian form $[\cdot, \cdot]$ by the formula $[x, y] = \bar{x}^t H y$. Let W be a complex subspace of $\mathbb{C}^n$ such that $[w_1, w_2] = 0$ for all $w_1, w_2 \in W$. Show that $\dim W \leq n/2$. Also give an example of an H for which $\dim W = n/2$ if n is even and $\dim W = (n - 1)/2$ if n is odd.

F12-8

- (35) Let $x_1, \dots, x_k$ be vectors in an inner product space V . Show that the $k \times k$ matrix $G(x_1, \dots, x_k)$ whose ij entry is (x_i, x_j) is self-adjoint and that all its eigenvalues are non-negative.

- (36) Show that the three subsets of $\text{Hom}(V, V)$ defined by

$$M_1 = \text{span } I,$$

$$M_2 = \{L : L \text{ is skew-adjoint}\},$$

$$M_3 = \{L : \text{tr } L = 0 \text{ and } L \text{ is self-adjoint}\},$$

are subspaces over $\mathbb{R}$, are mutually orthogonal with respect to the real inner product $(L, K) = \text{Re}(\text{tr}(L^*K))$, and yield a direct sum decomposition of $\text{Hom}(V, V)$.

- (37) Suppose that $L : V \rightarrow V$ is normal. Without invoking the spectral theorem, show the following.

(a) $\ker(L) = \ker(L^k)$ for any $k \geq 1$.

(b) $\text{im}(L) = \text{im}(L^k)$ for any $k \geq 1$.

(c) $\ker(L - \lambda I) = \ker((L - \lambda I)^k)$ for any $k \geq 1$.

(d) Show that the minimal polynomial of L has no multiple roots.

- (38) Let $L : V \rightarrow V$ be a linear operator on an inner product space that preserves right angles (i.e. $(L(x), L(y)) = 0$ if $(x, y) = 0$).

(a) Show that if $\|x\| = \|y\|$ and $(x, y) = 0$ then $\|L(x)\| = \|L(y)\|$.

(b) Show that $L = \lambda U$, where U is an isometry.

- (39) Let V be a real inner product space and $F : V \rightarrow V$ a bijective map that preserves distances, i.e. for all $x, y \in V$ we have

$$\|F(x) - F(y)\| = \|x - y\|.$$

(a) Show that $G(x) = F(x) - F(0)$ also preserves distances and that $G(0) = 0$.

(b) Show that $\|G(x)\| = \|x\|$ for all $x \in V$.

(c) Show that $(G(x), G(y)) = (x, y)$ for all $x, y \in V$.

(d) If $e_1, \dots, e_n$ is an orthonormal basis, show that $G(e_1), \dots, G(e_n)$ is an orthonormal basis.

(e) Show that

$$G(x) = (x, e_1)G(e_1) + \dots + (x, e_n)G(e_n)$$

and conclude that G is linear.

(f) Conclude that $F(x) = L(x) + F(0)$ for some linear isometry L .

- (40) Consider the Killing Form on $\text{Hom}(V, V)$, where $2 \leq \dim V < \infty$, defined by

$$\mathbb{K}(L, K) = \text{tr } L \text{tr } K - \text{tr}(LK).$$

(a) Show that $\mathbb{K}(L, K) = \mathbb{K}(K, L)$.

(b) Show that $K \mapsto \mathbb{K}(L, K)$ is linear.

(c) Assume in addition that V is an inner product space. Show that $\mathbb{K}(L, L) > 0$ if L is skew-adjoint and nonzero. Show that $\mathbb{K}(L, L) < 0$ if L is self-adjoint and nonzero.

(d) Show that $\mathbb{K}$ is nondegenerate: if $L \neq 0$ then there exists a K such that $\mathbb{K}(L, K) \neq 0$.