

LINEAR ALGEBRA BOOT CAMP

WEEK 3: INNER PRODUCT SPACES

Unless otherwise stated, all vector spaces in this worksheet are finite dimensional and the scalar field $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 1. An inner product on a vector space V over $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$) is an $\mathbb{F}$ -valued pairing (x, y) for $x, y \in V$, i.e. a map $(x, y) : V \rightarrow \mathbb{F}$ that satisfies

- (1) $(x, x) \geq 0$ with equality only if $x = 0$.
- (2) $(x, y) = \overline{(y, x)}$.
- (3) For each $y \in V$, the map $x \mapsto (x, y)$ is linear.

The length, or norm, of a vector is $\|x\| = \sqrt{(x, x)}$. Two vectors x, y are orthogonal if $(x, y) = 0$.

Lemma 2 (Pythagorean Theorem). If x, y are orthogonal then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Warmup. I should have put this on last week's worksheet.

- (1) Suppose A, B are invertible $n \times n$ matrices with complex entries such that $ABA^{-1} = B^3$. Show that all the eigenvalues of B are roots of unity.

F16-1

Projections, Cauchy-Schwarz and the Triangle Inequality.

Definition 3. The orthogonal projection of x onto y ($\neq 0$) is

$$\text{proj}_y(x) = \left(x, \frac{y}{\|y\|} \right) \frac{y}{\|y\|} = \frac{(x, y)}{(y, y)} y.$$

- (2) Prove the following properties of orthogonal projections:
 - (a) The map $x \mapsto \text{proj}_y(x)$ is a projection (recall: $E^2 = E$).
 - (b) The vectors $\text{proj}_y(x)$ and $x - \text{proj}_y(x)$ are orthogonal. In particular,

$$\|x\|^2 = \|x - \text{proj}_y(x)\|^2 + \|\text{proj}_y(x)\|^2.$$

- (c) $\|\text{proj}_y(x)\| \leq \|x\|$.

- (3) Use the previous exercise to prove the Cauchy-Schwarz Inequality

$$|(x, y)| \leq \|x\| \|y\|.$$

F02-9, W06-7

- (4) Prove the Cauchy-Schwarz Inequality again by expanding the right-hand side of

$$0 \leq \left\| x - \frac{(x, y)}{\|y\|^2} y \right\|^2.$$

- (5) Use Cauchy-Schwarz to prove the Triangle Inequality

$$\|x + y\| \leq \|x\| + \|y\|.$$

- (6) Consider the least squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|,$$

where $A \in \text{Mat}_{m \times n}(\mathbb{R})$. Prove that if x and y are both minimizers then $x - y \in \ker(A)$.

F08-12, F11-12, F12-9, S14-5

Orthonormal Bases.

Definition 4. A collection of vectors $\{e_j\}$ is orthogonal if $(e_i, e_j) = 0$ for $i \neq j$. If in addition we have $(e_i, e_j) = \delta_{ij}$ then we call the collection orthonormal.

- (7) Suppose $e_1, \dots, e_n$ are orthonormal. Prove that $e_1, \dots, e_n$ are linearly independent and that any $x \in \text{span}\{e_1, \dots, e_n\}$ has the expansion

$$x = (x, e_1)e_1 + \dots + (x, e_n)e_n.$$

- (8) Figure out how Gram-Schmidt works (let's be honest, nobody remembers this); that is, given a linearly independent set $\{x_1, \dots, x_n\}$ in an inner product space V , give a procedure for constructing an orthonormal collection $\{e_1, \dots, e_m\}$ such that

$$\begin{aligned} \text{span}\{x_1\} &= \text{span}\{e_1\}, \\ \text{span}\{x_1, x_2\} &= \text{span}\{e_1, e_2\}, \\ &\vdots \\ \text{span}\{x_1, \dots, x_n\} &= \text{span}\{e_1, \dots, e_n\}. \end{aligned}$$

- (9) Let $V = \{f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{C}\}$ be the complex vector space of polynomials in the variable x of degree at most 3.

(a) Show that V is an inner product space with

$$(f, g) = \int_{-1}^1 f(t) \overline{g(t)} dt.$$

(b) Find an orthonormal basis for V .

F17-1

- (10) Let $f : V \rightarrow \mathbb{C}$ be a linear functional. Show that there exists a vector $w \in V$ such that $f(v) = (v, w)$ for all $v \in V$.

W02-10

- (11) Let V, W be inner product spaces over $\mathbb{C}$ such that $\dim V \leq \dim W$. Prove that there is a linear transformation $T : V \rightarrow W$ such that $(Tv, Tv') = (v, v')$ for all $v, v' \in V$.

F10-7

- (12) Let $u_1, \dots, u_n$ be an orthonormal basis of $\mathbb{R}^n$ and let $y_1, \dots, y_n$ be a collection of vectors in $\mathbb{R}^n$ satisfying $\sum_i \|y_i\|^2 < 1$. Prove that the vectors $u_1 + y_1, \dots, u_n + y_n$ are linearly independent.

S10-1

The Operator Norm and the Exponential of a Matrix.

Definition 5. The operator norm for a linear map between inner product spaces $L : V \rightarrow W$ is

$$\|L\| = \sup_{\|x\|=1} \|L(x)\|.$$

- (13) Compute the (operator) norm of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}.$$

(For now, compute this directly from the definition; later we'll be more clever about it.)

S09-2

(14) Prove that $\|L(x)\| \leq \|L\| \|x\|$ for all $x \in V$ and that $\|L\| < \infty$.

(15) Let L, K be linear operators. Prove the following properties of the operator norm (the associated vector spaces will change in each part, but it should be clear from context).

- (a) If λ is an eigenvalue for L , then $|\lambda| \leq \|L\|$.
- (b) $\|K \circ L\| \leq \|K\| \|L\|$.
- (c) $\|L + K\| \leq \|L\| + \|K\|$.
- (d) If $A = (a_{ij})$ is a matrix representation for L with respect to an orthonormal basis then $|a_{ij}| \leq \|L\|$.

(16) Let A be a square matrix, and define the matrix exponential as

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Using the properties of the operator norm, prove the following.

- (a) $\exp(A)$ is convergent (that is, each entry in $\exp(A)$ is a convergent series).
- (b) If $AB = BA$, then $\exp(A + B) = \exp(A) \exp(B)$. (This is kind of tricky; use that the usual binomial expansion holds because A and B commute, and estimate the difference between the partial sums of the left- and right-hand sides.)
- F15-3 (Also find an example of noncommuting matrices where this fails)
- (c) If B is invertible, then $\exp(B^{-1}AB) = B^{-1} \exp(A) B$.

(17) Let $f(t) = \exp(At)$. Using the limit definition of the derivative and the properties from the previous problem, prove that $f'(t) = A \exp(At)$. Conclude that $\exp(At)x_0$ solves the initial value problem $x'(t) = Ax(t)$, $x(0) = x_0$.

~F09-8

(18) Suppose that $e^A = 1 + A + A^2$. Prove or disprove: A is a zero matrix.

S18-3

(19) Compute e^{At} when

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}.$$

S09-5 *Hint from Nick:* There is an “algorithmic” way to approach this problem which uses Jordan form. It’s fine. But for this specific problem, after you find the characteristic polynomial, look closely at it before proceeding.

- (20) Let A be an invertible $n \times n$ matrix with entries in $\mathbb{C}$. Suppose that the set $\{\|A^n\| : n \in \mathbb{Z}\}$ is bounded. Show that A is diagonalizable.

F12-7

- (21) Let $A \in M_n(\mathbb{R})$ and let $B = A^t A$. Next week we'll show that there is a basis for $\mathbb{R}^n$ consisting of eigenvectors for B . Assuming this fact for now, show that $\|A\|^2$ is equal to the largest eigenvalue of B .

- (22) Suppose

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Find the smallest possible constant $C > 0$ such that $\|e^A x\| \leq C \|x\|$ for all $x \in \mathbb{R}^2$.

F16-2

Orthogonal Complements and Projections.

Definition 6. Suppose that $M \subset V$ is a subspace. The orthogonal complement to M is

$$M^\perp = \{x \in V : (x, z) = 0 \text{ for all } z \in M\}.$$

Let $e_1, \dots, e_m$ be an orthonormal basis for M , and define $E : V \rightarrow V$ by

$$E(x) = (x, e_1)e_1 + \dots (x, e_m)e_m.$$

Then $E(V) \subset M$ and E fixes M pointwise. So E is a projection ($E^2 = E$) onto M with $\ker E = M^\perp$. By exercise (20) from Week 1, we have $V = M \oplus M^\perp$.

Definition 7. The projection E defined above is called the orthogonal projection onto M and we write it $\text{proj}_M : V \rightarrow V$.

- (23) If $W_1, W_2 \subset V$ are subspaces of an inner product space V , prove that

$$(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp.$$

F10-8

- (24) Let $x \in V$ and let $M \subset V$ be a subspace. Use the Pythagorean Theorem to show that $\text{proj}_M(x)$ is the vector in M that is closest to x .

~ S02-3

- (25) Suppose that $E : V \rightarrow V$ is a projection onto M . Prove that $E = \text{proj}_M$ if and only if $\|E(x)\| \leq \|x\|$ for all $x \in V$. (It may be useful to use that $E = \text{proj}_M$ if and only if $\text{im}(E)^\perp = \ker E$).

- (26) Consider the system $Ax = b$, where A is a 3×4 matrix with real entries. Assuming there is at least one solution to the system, show that the solution with minimal length is $A^t u$, where u is a solution to the system $AA^t u = b$.

- (27) Among all solutions to the system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 7 \\ -2 & -1 & 1 & 3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix}$$

find the solution with minimal length.

F14-7 *Hint from Nick:* row reduce first! The answer is unpleasant.

Adjoint Maps.

- (28) Let $L : V \rightarrow W$ be a linear map between inner product spaces.
- (a) Construct a map $L^* : W \rightarrow V$ (called the adjoint) which satisfies $(Lx, y) = (x, L^*y)$ for all $x \in V, y \in W$. (One way to do this: let $e_1, \dots, e_m$ be an orthonormal basis for V . If we know what (L^*y, e_j) should be for all j then we can define $L^*y = \sum (L^*y, e_j)e_j$.)
 - (b) Show that the adjoint is uniquely defined.
 - (c) Show that if matrices are written relative to orthonormal bases of V and W then the matrix of L^* is the conjugate transpose of the matrix of L (i.e. $[L^*] = \overline{[L]}^t$).

~W06-8, S13-8ab

- (29) Let A be a 3×3 real matrix with $A^t A = I = AA^t$ and $\det(A) = 1$. Prove that the characteristic polynomial for A has 1 as a root. (You may use that the determinant is the product of the eigenvalues.)

S03-9, S06-8, F09-9

- (30) Let $T : V \rightarrow V$ be a linear operator on a real inner product space. Prove that the following are equivalent.
- (a) $(Tx, Ty) = (x, y)$ for all $x, y \in V$.
 - (b) $\|T(x)\| = \|x\|$ for all $x \in V$.
 - (c) $T^*T = I$.
 - (d) $TT^* = I$.

S04-9

- (31) Let A be a symmetric $n \times n$ real matrix, $n \geq 4$, and let $v_1, \dots, v_4 \in \mathbb{R}^n$ be nonzero vectors. Suppose $Av_k = (2k-1)v_k$ for all $k = 1, \dots, 4$. Prove that $v_1 + 2v_2$ is orthogonal to $3v_3 + 4v_4$.

S16-12

- (32) Prove the Fredholm Alternative:

$$\begin{aligned}\ker(L) &= \operatorname{im}(L^*)^\perp, \\ \ker(L^*) &= \operatorname{im}(L)^\perp, \\ \ker(L)^\perp &= \operatorname{im}(L^*), \\ \ker(L^*)^\perp &= \operatorname{im}(L).\end{aligned}$$

You can (and should) use that $L^{**} = L$ and $M^{\perp\perp} = M$ to reduce these to one equation.

F03-8b, ~S11-5, ~S13-8c

- (33) Use the Fredholm Alternative to prove the Rank Theorem again: $\operatorname{rank}(L) = \operatorname{rank}(L^*)$ (row rank equals column rank).

~S13-8d

- (34) Let M be an $n \times n$ real matrix. Prove that M and MM^t have the same image.

S17-1

- (35) Let $A : V \rightarrow W$ be a linear map between real inner product spaces, and fix $w \in W$. Show that the elements $v \in V$ for which $\|Av - w\|$ is minimal are exactly the solutions to the equation $A^*Ax = A^*w$.

S07-1, S11-6

- (36) (a) Show that $(A, B) = \operatorname{tr}(AB^t)$ defines an inner product on $M_n(\mathbb{R})$.
(b) For $n = 2$, find an orthonormal basis for $M_2(\mathbb{R})$.

S17-6

- (c) Fix $C \in M_n(\mathbb{R})$ and define

$$\Phi_C : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R}) \quad \text{by} \quad \Phi_C(A) = CA - AC.$$

Compute the adjoint of Φ_C . Check that when C is symmetric, Φ_C is self-adjoint.

- (d) Show that for every C the map Φ_C is not onto.

S09-8

Extra Problems.

- (37) Let A be an $n \times n$ real symmetric matrix and let $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit sphere of $\mathbb{R}^n$. Let $x \in S$ be such that

$$(Ax, x) = \sup_S (Ay, y).$$

(By compactness, such an x exists.)

- (a) Prove that if $(x, y) = 0$ then $(Ax, y) = 0$. *Hint:* Expand

$$(A(x + \varepsilon y), x + \varepsilon y).$$

- (b) Use (a) to prove that x is an eigenvector for A .

- (c) Use induction to prove that $\mathbb{R}^n$ has an orthonormal basis of eigenvectors for A .

S08-12

- (38) Let A be a real $n \times n$ symmetric matrix and let $\lambda_1 \geq \dots \geq \lambda_n$ be the eigenvalues of A . Prove that

$$\lambda_k = \max_{U: \dim U = k} \min_{x \in U: \|x\|=1} (Ax, x),$$

where the maximum is taken over all k -dimensional subspaces of $\mathbb{R}^n$.

S10-2

- (39) A real symmetric $n \times n$ matrix A is a reflection if $A^2 = I$ and $\text{rank}(A - I) = 1$. Given distinct unit vectors $x, y \in \mathbb{R}^n$ with $x \neq -y$, show that there is a reflection A with $Ax = y$ and $Ay = x$. Moreover, show that the reflection A with these properties is unique.

S09-9

- (40) An orthogonal projection is an endomorphism P that satisfies $P^2 = P$ and $\text{im}(P) = \ker(P)^\perp$. Suppose $V = \mathbb{R}^3$ and P is an orthogonal projection with diagonal entries $p_{11} = 2/3$, $p_{22} = 1/2$, and $p_{33} = 5/6$. Find all matrices that P could be.

F13-8

- (41) Assume $V = M \oplus M^\perp$ and that $L : V \rightarrow V$ is linear. Show that both M and $M^\perp$ are L -invariant if and only if $\text{proj}_M \circ L = L \circ \text{proj}_M$.

- (42) Let $M, N \subset V$ be subspaces of a finite-dimensional inner product space. For this problem, we will use angle brackets to denote the inner product on V : $\langle \cdot, \cdot \rangle_V$. There is a natural inner product on $M \times N$ given by

$$\langle (x, x'), (y, y') \rangle_{M \times N} = \langle x, y \rangle_V + \langle x', y' \rangle_V.$$

Consider the linear map $L : M \times N \rightarrow V$ given by $L(x, x') = x - x'$.

- (a) Show that $L^*(v) = (\text{proj}_M v, -\text{proj}_N v)$.
 (b) Show that

$$\begin{aligned} \ker(L^*) &= M^\perp \cap N^\perp, \\ \text{im}(L) &= M + N. \end{aligned}$$

- (c) Using the Fredholm alternative, show that

$$(M + N)^\perp = M^\perp \cap N^\perp.$$

- (d) Replace M and N by $M^\perp$ and $N^\perp$ and conclude that

$$(M \cap N)^\perp = M^\perp + N^\perp.$$

- (43) Prove Hölder's Inequality for vectors in $\mathbb{C}^n$:

Theorem 8 (Hölder's Inequality). If $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{C}^n$ and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\sum_{k=1}^n |a_k b_k| \leq \|a\|_p \|b\|_q,$$

where

$$\|a\|_p = \left(\sum_{k=1}^n |a_k|^p \right)^{1/p}.$$

(Note that Cauchy-Schwarz is the case $p = q = 2$.) *Hint*: first prove Young's inequality: if $a, b \geq 0$ and $1/p + 1/q = 1$ then

$$ab \leq \frac{a^p}{p} + \frac{a^q}{q}.$$

- (44) Prove Enflo's inequality:

Theorem 9. Let $x_1, \dots, x_n$ and $y_1, \dots, y_n \in V$. Then

$$\left(\sum_{i,j=1}^n |(x_i, y_j)|^2 \right)^2 \leq \left(\sum_{i,j=1}^n |(x_i, x_j)|^2 \right) \left(\sum_{i,j=1}^n |(y_i, y_j)|^2 \right).$$

Hint: Use an orthonormal basis and start expanding on the left-hand side.

- (45) Let $M \subset V$ be a subspace with orthonormal basis $e_1, \dots, e_m$. Together with the formula

$$\text{proj}_M(x) = (x, e_1)e_1 + \dots + (x, e_m)e_m,$$

the inequality $\|\text{proj}_M(x)\| \leq \|x\|$ translates to the Bessel inequality

$$|(x, e_1)|^2 + \dots + |(x, e_m)|^2 \leq \|x\|^2.$$

Prove Selberg's generalization (here the y_i are arbitrary vectors):

$$\sum_{i=1}^n \frac{|(x, y_i)|^2}{\sum_{j=1}^n |(y_i, y_j)|} \leq \|x\|^2.$$

Hint: expand the nonnegative quantity

$$\left\| x - \sum_{i=1}^n \frac{(x, y_i)}{\sum_{j=1}^n |(y_i, y_j)|} y_i \right\|^2.$$

- (46) Let $C^0([0, 1], \mathbb{R})$ denote the space of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ and let V be a subspace. Consider the linear functionals $f_{t_0}(x) := x(t_0)$ and $f_y(x) = \int_0^1 x(t)y(t) dt$.

- (a) Suppose that V is finite dimensional. Show that $f_{t_0}|_V = f_y|_V$ for some $y \in V$.
- (b) Suppose that $V = P^2$, the space of polynomials of degree ≤ 2 . Find an explicit $y \in V$ as in part (a).
- (c) Suppose that $V = C^0([0, 1], \mathbb{R})$. Prove that there is no $y \in C^0([0, 1], \mathbb{R})$ such that $f_{t_0} = f_y$.

The illusory “function” δ_{t_0} invented by Dirac to solve this problem is called Dirac's δ -function. It is defined as

$$\delta_{t_0}(t) = \begin{cases} 0 & \text{if } t \neq t_0, \\ \infty & \text{if } t = t_0, \end{cases}$$

$$\int_0^1 \delta_{t_0} dt = 1,$$

so as to give the impression that

$$\int_0^1 x(t)\delta_{t_0}(t) dt = x(t_0).$$