LINEAR ALGEBRA BOOT CAMP
WEEK 1: THE BASICS

Unless otherwise stated, all vector spaces in this worksheet are finite dimensional and the scalar field $\mathbb{F}$ has characteristic zero.

The following are facts (in no particular order) that you may assume for the purposes of this worksheet (and, as far as I can tell, for the basic exam).

Theorem 1. Every vector space has a basis.

Theorem 2 (Steinitz Replacement)**.** Let $\{y_1, \ldots, y_m\}$ be linearly independent and suppose that $V = \text{span}\{x_1, \ldots, x_n\}$. Then $m \leq n$ and V has a basis of the form $y_1, \ldots, y_m, x_{i_1}, \ldots, x_{i_\ell}$ where $\ell \leq n - m$.

Lemma 3. Two vector spaces V, W are isomorphic if and only if there is a bijective (one-to-one and onto) linear transformation $T : V \to W$.

Lemma 4. A linear transformation $T : V \to W$ is one-to-one if and only if $\ker(T) = \{0\}$.

Lemma 5. If A and B are square matrices then $\text{tr}(AB) = \text{tr}(BA)$.

Lemma 6. The transpose of a matrix satisfies $(AB)^T = B^T A^T$.

Lemma 7. Suppose V is finite dimensional with basis $x_1, \ldots, x_n$. Then the dual space V' has basis $\phi_1, \ldots, \phi_n$, where $\phi_i(x_j) = \delta_{ij}$. In particular, $\dim(V') = \dim(V)$.

Lemma 8. If $M, N \subset V$ are subspaces, then the sets

$$M + N = \{m + n : m \in M, n \in N\}$$
$$M \cap N = \{x \in V : x \in M \text{ and } x \in N\}$$

are subspaces. The span of a subset $X \subset V$, denoted $\text{span}(X)$, is the smallest subspace of V containing X and we have

$$M + N = \text{span}(M \cup N).$$

The Rank-Nullity Theorem and "pick a basis".

(1) Prove the Rank-Nullity Theorem.

Theorem 9 (The Rank-Nullity Theorem)**.** If $T : V \to W$ is a linear transformation then

$$\dim(\mathrm{im}(T)) + \dim(\ker(T)) = \dim(V).$$

F03-8a, F08-6 *Hint:* pick a basis for $\ker T$, use the Steinitz Replacement Theorem to extend it to a basis for V. Which vectors form a basis for $\operatorname{im} T$?

Now prove the following corollary.

Corollary 10. If $\dim V = \dim W$ then a linear transformation $T : V \to W$ is an isomorphism if and only if $\ker(T) = \{0\}$.

(2) Let V, W, Z be n-dimensional vector spaces and $T : V \to W$ and $U : W \to Z$ be linear transformations. Prove that if the composite $UT : V \to Z$ is invertible, then both T and U are invertible (don't use determinants!).
S07-2

(3) Let V, W, X be vector spaces and let $T : V \to W$ and $S : W \to X$ be linear transformations. Prove *Sylvester's Rank Inequality*:

$$\operatorname{rank}(T) + \operatorname{rank}(S) - \dim(W) \leq \operatorname{rank}(S \circ T) \leq \min\{\operatorname{rank}(T), \operatorname{rank}(S)\}.$$

$\boxed{\text{W02-8, S14-2}}$

(4) Let A, B be two 4×5 matrices of rank 3 and let $C = A^T B$. Find all possible values for the rank of C. That is, for each possible value, find an explicit example of such matrices. Then prove that all other values are impossible.

$\boxed{\text{F15-7}}$ *Hint:* use Sylvester's Rank Inequality.

(5) Suppose $W_1, W_2 \subset V$ are subspaces.
   (a) Show that
$$\dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(\operatorname{span}(W_1, W_2)),$$
   where $\operatorname{span}(W_1, W_2)$ is the smallest subspace that contains both W_1 and W_2.
   (b) Let $n = \dim V$. Use part (a) to prove that if $k < n$ then an intersection of k subspaces of dimension $n-1$ always has dimension at least $n-k$. *Hint:* induction on k.

$\boxed{\text{F05-9}}$

Extra Problems

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(6) Let S and S' denote subsets of a vector space V. Prove or disprove:
   (a) $\operatorname{span}(S) \cap \operatorname{span}(S') = \operatorname{span}(S \cap S')$
   (b) $\operatorname{span}(S) + \operatorname{span}(S') = \operatorname{span}(S \cup S')$.

$\boxed{\text{F10-5}}$

(7) Suppose $T : V \to W$ and $S : W \to U$ are linear transformations such that T is one-to-one, S is onto, and $S \circ T = 0$. Prove that $\operatorname{im}(T) \subset \ker(S)$ and that
$$-\dim(V) + \dim(W) - \dim(U) = \dim(\ker(S)/\operatorname{im}(T)).$$

$\boxed{\text{F11-11}}$

(8) Let $V = \mathbb{R}^n$ and let $U_1, U_2, W_1, W_2 \subset V$ be subspaces of V of dimension d such that $\dim(U_1 \cap W_1) = \dim(U_2 \cap W_2) = \ell$ for some $\ell \leq d \leq n$. Prove that there exists a linear operator $T : V \to V$ such that $T(U_1) = U_2$ and $T(W_1) = W_2$.

$\boxed{\text{S15-9}}$

Matrices.

(9) Prove the Rank Theorem.

Theorem 11 (Rank Theorem)**.** Suppose that A is an $n \times m$ matrix. Then the row rank of A (the maximal number of linearly independent rows) is equal to the column rank of A (the maximal number of linearly independent columns).

$\boxed{\text{F03-8c, F05-7}}$ *Hint:* If $x_1, \ldots, x_r$ is a basis for the column space of A (think of them as $n \times 1$ column vectors), then $A = [x_1 \ \ldots \ x_r]B$ for some $r \times m$ coordinate matrix B. By taking transposes, show that this implies that row rank$(A) \leq$ col rank(A). Now apply the same argument to A^T for the reverse inequality.

(10) Let $t_1, \dots, t_n \in \mathbb{F}$ be distinct. Prove that the vectors

$$\begin{bmatrix} 1 \\ t_1 \\ \vdots \\ t_1^{n-1} \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ t_n \\ \vdots \\ t_n^{n-1} \end{bmatrix}$$

form a basis for $\mathbb{F}^n$. *Hint:* compute the rank of the matrix whose columns are the vectors above using the fact that row rank equals column rank.

(11) Suppose that $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ are distinct and that

$$\sum_{k=1}^{n} a_k e^{\alpha_k t} = 0 \qquad \text{for all } t \in (-1, 1).$$

Prove that $a_k = 0$ for all k.

$\boxed{\text{F09-5}}$ *Hint:* construct a vector-valued function from $(-1, 1)$ to $\mathbb{C}^n$ using repeated differentiation. Then use the previous exercise.

Extra Problem

(12) Let $\mathcal{A} = M_n(\mathbb{C})$. We say that $\mathcal{I} \subset \mathcal{A}$ is a two-sided ideal if
   (i) for all $A, B \in \mathcal{I}$, $A + B \in \mathcal{I}$, and
   (ii) for all $A \in \mathcal{A}$, $B \in \mathcal{I}$, the matrices AB and BA both belong to $\mathcal{I}$.
   Prove that $\mathcal{A}$ has no nontrivial (i.e. not equal to $\{0\}$ or $\mathcal{A}$) two-sided ideals.
   $\boxed{\text{S05-4}}$ *Hint:* suppose $0 \neq X \in \mathcal{I}$. Let E_{ij} denote the matrix with a 1 in the ith row, jth column and 0s elsewhere. By multiplying on the right/left by E_{ij} and diagonal matrices, show that $E_{ii} \in \mathcal{I}$ for each i, and therefore $I_n \in \mathcal{I}$.

The dual space and the annihilator of a subspace. If V is a vector space then

$$V' = \mathrm{Hom}(V, \mathbb{F}) := \{f : V \to \mathbb{F} : f \text{ is linear}\}$$

is the dual space of V. If $U \subset V$ is a subset then

$$U^0 := \left\{f \in V' : f(u) = 0 \text{ for all } u \in U\right\}$$

is the annihilator of U. This is sometimes denoted $U^{\perp}$.

(13) Let V be a finite dimensional vector space and let $W \subset V$ be a subspace. Prove that

$$\dim(V) = \dim(W) + \dim(W^0).$$

   $\boxed{\text{S02-8}}$

(14) Let $T : V \to W$ be a linear transformation of vector spaces. The transpose of T is the map $T^t : W' \to V'$ defined by

$$T^t(f) = f \circ T.$$

   Prove the following.
   (a) $\mathrm{im}(T)^0 = \ker(T^t)$
   (b) $\dim(\mathrm{im}(T)) = \dim(\mathrm{im}(T^t))$ (this is another way to prove that row rank = column rank)
   $\boxed{\text{F01-7, F02-7}}$

(15) Let V be a vector space and let $W_1, W_2 \subset V$ be subspaces. Prove that
$$W_1^0 \cap W_2^0 = (W_1 + W_2)^0 \quad \text{and}$$
$$(W_1 \cap W_2)^0 = W_1^0 + W_2^0.$$

$\boxed{\text{S03-7, S04-7}}$ *Hint:* the first equation and the inclusion $\supset$ of the second are straightforward. To prove $\subset$ in the second equation, compute the dimension of each side using several previous exercises and the first equation.

(16) Let $M \subset V$ be a subspace. Show that we have linear maps
$$M^0 \xrightarrow{i} V' \xrightarrow{\pi} M',$$
where i is one-to-one, π is onto, and $\mathrm{im}(i) = \ker(\pi)$. Conclude that $V' \cong M^0 \times M'$.

(17) Fix $n+1$ points $x_0 < x_1 < \ldots < x_n$ in $[a,b]$. Define linear functions ℓ_j on $\mathbb{P}^n$, the space of polynomials of degree less than or equal to n, by setting

$$\ell_j(p) = p(x_j), \qquad j = 0, 1, \ldots, n.$$

(a) Prove that $\{\ell_j : j = 0, 1, \ldots, n\}$ is linearly independent.
(b) Show that there are unique coefficients c_j such that

$$\int_a^b p(x)\,dx = \sum_{j=0}^{n} c_j \ell_j(p)$$

for all $p \in \mathbb{P}^n$.

$\boxed{\text{F07-12}}$

Extra Problem

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(18) Let V be a vector space. For a set $U \subset V'$, let

$$^{\perp}U = \{v \in V : \phi(v) = 0 \text{ for all } \phi \in U\}.$$

(a) Show that for any subset $W \subset V$ we have $^{\perp}(W^{\perp}) = \operatorname{span}(W)$.
(Here the span of W is the smallest subspace of V containing W.)
(b) Let $W \subset V$ be a linear subspace. Give (with proof) an explicit isomorphism between $(V/W)'$ and $W^{\perp}$.

$\boxed{\text{S05-2}}$

No machinery required.

(19) Let $T : V \to V$ be a linear transformation such that Tv and v are linearly dependent for each $v \in V$. Show that T must be a scalar multiple of the identity.
$\boxed{\text{F07-3}}$

(20) Assume V is a vector space over $\mathbb{Q}$ and that $T : V \to V$ is a linear transformation such that $T^2 = T$. Prove that every vector $v \in V$ can be written uniquely as $v = v_1 + v_2$, where $T(v_1) = v_1$ and $T(v_2) = 0$.
$\boxed{\text{S08-8}}$

(21) Let $\alpha_1, \ldots, \alpha_n$ be nonzero complex numbers, and let

$$V = \left\{ \sum_{i=1}^{n} r_i \alpha_i : r_i \in \mathbb{Q} \right\}$$

be the vector subspace of $\mathbb{C}$ (thought of as a $\mathbb{Q}$-vector space) spanned by them over $\mathbb{Q}$. Let β be a complex number such that $\beta V \subset V$, where $\beta V = \{\beta v : v \in V\}$. Show that β is the root of a degree n polynomial with rational coefficients.
$\boxed{\text{F16-5}}$

Extra Problems

(22) Let $V = \mathbb{R}^n$ and let $T : V \to V$ be a linear transformation. For $\lambda \in \mathbb{C}$, the subspace

$$V(\lambda) = \{v \in V : (T - \lambda I)^N v = 0 \text{ for some } N \geq 1\}$$

is called a generalized eigenspace.

(a) Prove that there exists a fixed number M such that

$$V(\lambda) = \ker((T - \lambda I)^M).$$

Hint: prove that for any T, (1) $\ker(T) \subset \ker(T^2) \subset \dots$ and (2) if for some $m \geq 1$ we have $\ker(T^m) = \ker(T^{m+1})$ then $\ker(T^m) = \ker(T^{m+k})$ for all $k \geq 0$.

(b) Prove that if $\lambda \neq \mu$ then $V(\lambda) \cap V(\mu) = \{0\}$.
Hint: use the following equation by raising both sides to a high power:

$$\frac{T - \lambda I}{\mu - \lambda} + \frac{T - \mu I}{\lambda - \mu} = I.$$

F04-10

(23) Consider the space $\mathcal{S}$ of infinite sequences (a_n) of real numbers endowed with the standard operations of addition and scalar multiplication: $(a_n) + (b_n) = (a_n + b_n)$ and $c(a_n) = (ca_n)$. Let A and B be fixed real numbers. Prove that the set of solutions to the linear recursion

$$x_{n+2} = Ax_{n+1} + Bx_n$$

is a linear subspace of $\mathcal{S}$ of dimension 2.

S07-3

(24) Let $v_1 = (0, 1, x)$, $v_2 = (1, x, 1)$, and $v_3 = (x, 1, 0)$. Find all $x \in \mathbb{R}$ such that $\{v_1, v_2, v_3\}$ are linearly independent over $\mathbb{R}$. Is the answer the same over $\mathbb{Q}$?

S16-9

(25) A linear transformation $E : V \to V$ is called a *projection* onto a subspace M if $E^2 = E$ and $\mathrm{im}(E) = M$. Let $L : V \to V$ be a linear transformation and $M \subset V$ a subspace. Show:

(a) If E is a projection onto M and $ELE = LE$ then M is invariant under L.

(b) If M is invariant under L then $ELE = LE$ for all projections E onto M.

Trace of a matrix.

(26) Suppose that $A, B \in M_n(\mathbb{C})$ satisfy $AB - BA = A$. Show that A is not invertible.

$\boxed{\text{S14-3}}$

(27) Given $n \geq 1$, let $\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ denote the trace of an $n \times n$ matrix.

(a) Determine a basis for the kernel of tr.

(b) For $X \in M_n(\mathbb{C})$ show that $\mathrm{tr}(X) = 0$ if and only if there exists an integer m and matrices A_j, B_j, $1 \leq j \leq m \in M_n(\mathbb{C})$ such that

$$X = \sum_{j=1}^{m} (A_j B_j - B_j A_j).$$

$\boxed{\text{S05-1}}$

Extra Problems

Affine subspaces.

(28) An affine subspace $A \subset V$ of a vector space is a subset such that

$$\text{if } \alpha_1 + \ldots + \alpha_n = 1 \text{ and } x_1, \ldots, x_n \in A \text{ then } \alpha_1 x_1 + \ldots + \alpha_n x_n \in A.$$

(a) Show that A is an affine subspace if and only if there is a point $x_0 \in V$ and a subspace $M \subset V$ such that

$$A = x_0 + M = \{x_0 + x : x \in M\}.$$

(b) Show that A is an affine subspace if and only if there is a subspace $M \subset V$ with the properties (1) if $x, y \in A$ then $x - y \in M$ and (2) if $x \in A$ and $z \in M$ then $x + z \in A$.

(c) Show that the subspaces from parts (a) and (b) are equal.

(29) Denote by G the set of real 4×4 upper triangular matrices with 1s on the diagonal. Fix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G.$$

Let $C \subset G$ be the set of matrices in G commuting with M.

(a) Prove that C is an affine subspace in the space $\mathbb{R}^{16}$ of all 4×4 matrices.
Hint: the characterization (a) of affine subspaces is natural here. Also, this is a brute-force computation that is not as bad as it looks.

(b) Find the dimension of C.

$\boxed{\text{S13-10}}$

Infinite dimensional vector spaces.

(30) Let V be a (not necessarily finite dimensional) vector space over $\mathbb{R}$ and let $V' = \mathrm{Hom}(V, \mathbb{R})$ be the dual space. Let $\mathcal{B} = \{e_i\}_{i \in I}$ be a basis for V. For each $i \in I$, define the dual forms $e_i^{\#} \in V'$ by the rule

$$e_i^{\#}(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

(a) Show that the vectors $\{e_i^{\#}\}_{i \in I}$ are linearly independent in V'.

(b) Give necessary and sufficient conditions on V for these vectors to form a basis of V'. Prove your claim.

$\boxed{\text{F17-4}}$

(31) Let V and W be two infinite-dimensional vector spaces over $\mathbb{C}$.

(a) Is $X = \{f \in \mathrm{Hom}(V, W) : f \text{ has finite rank}\}$ a subspace of $\mathrm{Hom}(V, W)$?

(b) Is $Y = \{f \in \mathrm{Hom}(V, W) : \ker(f) \text{ is finite dimensional}\}$ a subspace of $\mathrm{Hom}(V, W)$?

(c) What is the intersection $X \cap Y$?

$\boxed{\text{F17-5}}$

(32) Let $\mathbb{F}$ be the finite field with p elements, let V be an n-dimensional vector space over $\mathbb{F}$, and let $0 \le k \le n$. Compute the number of invertible linear maps $V \to V$. It is acceptable if your solution is a lengthy algebraic expression, as long as you explain why it is correct.
$\boxed{\text{S12-7}}$ Note: I think this is the only past exam question that is not in characteristic zero.

(33) Let $z_1, \dots, z_n$ be distinct complex numbers and for $1 \le j \le n$, let m_j be a nonnegative integer. Write $N + 1 = \sum_{j=0}^{n}(1 + m_j)$. Prove that, given any array of $N + 1$ complex numbers

$$c_{j,k}, \qquad 1 \le j \le n, 0 \le k \le m_j,$$

there is a unique polynomial $P(z)$ of degree at most N such that for all j, k,

$$P^{(k)}(z_j) = c_{j,k},$$

where $P^{(k)}$ denotes the k-th derivative.
$\boxed{\text{F13-7}}$

(34) Let A be an endomorphism of a vector space V of dimension n over a field $\mathbb{F}$. Show from first principles (i.e. do not use Jordan form or the Cayley-Hamilton theorem) that A satisfies a polynomial $P(x) \in \mathbb{F}[x]$ of degree at most n.
$\boxed{\text{F13-9}}$

(35) What is the largest number of 1s that an invertible 0-1 matrix of size $n \times n$ can have? You must show both that this number is possible and that no larger number is possible.
$\boxed{\text{F14-10}}$

(36) Two matrices A, B are called *commuting* if $AB = BA$. The *order* of a matrix A is the smallest integer $k > 0$ such that $A^k = 1$; if no such k exists, the order is defined to be infinite. Prove that there exist 10 distinct real 2×2 matrices which are all pairwise commuting and all of the same finite order.
$\boxed{\text{S15-11}}$

(37) Let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find necessary and sufficient conditions on a 3×3 matrix A with real entries so that the equation

$$A = R^T B R$$

admits a solution R of full rank. Here R must be a 2×3 matrix with real entries. You must prove that your conditions are necessary and sufficient.
$\boxed{\text{F16-3}}$