MATH 205A: ANALYTIC NUMBER THEORY
HOMEWORK 3

Attempt every problem. Submissions written in LaTeX will receive 5 bonus points.

(1) (30 points) Suppose that χ is a nonprincipal character modulo q. Since $\sum_{n(q)} \chi(n) = 0$, we have the bound

$$\sum_{n=M+1}^{M+N} \chi(n) \leq \phi(q)$$

for any M and N. It turns out that we can do quite a bit better than that. In this exercise we will prove the Polya-Vinogradov inequality

$$\left| \sum_{n=M+1}^{M+N} \chi(n) \right| < \sqrt{q}\, \log q$$

under the assumption that χ is a primitive character. This argument is worked out in detail in Section 23 of Davenport in case you get stuck, but try not to look at it; it's more fun to work it out yourself.

(a) Start by expressing $\chi(n)$ in terms of the additive character $e(an/q)$ (see the section of the notes on Gauss sums), then swap the order of summation and show that

$$\left| \sum_{n=M+1}^{M+N} \chi(n) \right| \leq q^{-\frac{1}{2}} \sum_{a=1}^{q-1} \frac{1}{\sin(\pi a/q)}.$$

(b) Suppose that f is a convex twice-differentiable function (its graph lies above the tangent line at any point). Prove that

$$f(\alpha) \leq \frac{1}{\delta} \int_{\alpha - \frac{1}{2}\delta}^{\alpha + \frac{1}{2}\delta} f(t)\, dt.$$

Apply this inequality with $f(\alpha) = (\sin \pi\alpha)^{-1}$ and $\delta = 1/q$ to prove that

$$\sum_{a=1}^{q-1} \frac{1}{\sin(\pi a/q)} \leq q \int_{\frac{1}{2q}}^{1-\frac{1}{2q}} \frac{dt}{\sin \pi t} = 2q \int_{\frac{1}{2q}}^{\frac{1}{2}} \frac{dt}{\sin \pi t}.$$

(c) Davenport uses the inequality $\sin \pi x \geq 2x$ for $x \in [0, 1/2]$ to finish the proof. That gives a nice clean answer, but we can do a bit better numerically with a bit more work. Prove that

$$\int_{\frac{1}{2q}}^{\frac{1}{2}} \frac{dt}{\sin \pi t} = \frac{1}{\pi} \log \cot\left(\frac{\pi}{4q}\right) < \frac{1}{\pi} \log\left(\frac{4q}{\pi}\right).$$

Conclude that

$$\left| \sum_{n=M+1}^{M+N} \chi(n) \right| < \frac{2}{\pi}\sqrt{q}\, \log q + \frac{2\log(4/\pi)}{\pi}\sqrt{q}.$$

(2) (10 points) The Möbius function $\mu(n)$ is the multiplicative function defined by $\mu(1) = 1$ and $\mu(p^k) = -1$ if $k = 1$ and 0 if $k \geq 2$.

(a) Prove that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)}.$$

(b) Using that $\frac{1}{\zeta(s)} \cdot \zeta(s) = 1$, prove that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(3) (30 points) Suppose that χ is a nonprincipal character modulo q which is induced by the primitive character χ' modulo q', and write $q = q'r$. Then

$$\sum_{n=M+1}^{M+N} \chi(n) = \sum_{\substack{n=M+1 \\ (n,r)=1}}^{M+N} \chi'(n) = \sum_{n=M+1}^{M+N} \chi'(n) \sum_{d|(n,r)} \mu(d).$$

(a) Prove that

$$\left| \sum_{n=M+1}^{M+N} \chi'(n) \sum_{d|(n,r)} \mu(d) \right| < d(r)\sqrt{q'} \log q',$$

where $d(r)$ is the number of divisors of r.

(b) By pairing up each divisor $d \leq \sqrt{r}$ with its complement $\frac{r}{d} \geq \sqrt{r}$, prove that

$$d(r) \leq 2\sqrt{r},$$

and conclude that the Polya-Vinogradov inequality (modified with an extra factor of 2 on the right-hand side) holds for all nonprincipal characters.

(4) (30 points) Suppose that χ is a primitive Dirichlet character modulo q. Applying partial summation to the Gauss sum

$$\tau(q) = \sum_{n \leq q} \chi(n) e(n/q)$$

prove that

$$\max_{t \leq q} \left| \sum_{n \leq t} \chi(n) \right| \geq \frac{1}{2\pi} \sqrt{q}.$$

This shows that the Polya-Vinogradov inequality is nearly best-possible.