MATH 205A: ANALYTIC NUMBER THEORY
HOMEWORK 2

This assignment is worth 100 points. You do not need to attempt every problem to earn all of the points. Submissions written in LaTeX will receive 5 bonus points.

(1) (15 points)

(a) Use Stirling's formula to prove that for bounded σ and $|t| \gg 1$,

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}.$$

You may find it helpful to expand $\arctan\left(\frac{y}{x}\right) = \frac{\pi}{2} - \frac{x}{y} + O\left(\frac{1}{y^3}\right)$.

(b) Prove that the entire function

$$f(s) = \frac{4^s\Gamma(s)\Gamma(s+1/2)}{\Gamma(2s)}$$

is constant, and that it equals $2\sqrt{\pi}$. This proves the duplication formula.

(2) (15 points)

(a) Prove that

$$\sum_{n\le x}\frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right).$$

(b) (Dirichlet's hyperbola method)
Let $d(n) = \sum_{d|n} 1$ denote the number of divisors of n. Then

$$\sum_{n\le X} d(n) = \sum_{n\le X}\sum_{d|n} 1 = \sum_{n\le X}\sum_{ab=n} 1 = \sum_{ab\le X} 1.$$

The latter sum counts the number of integer lattice points underneath the hyperbola $xy = X$. The hyperbola is symmetric about $y = x$ (i.e. across the point $(\sqrt{X}, \sqrt{X})$) so it makes sense to count only those a which are $\le \sqrt{X}$, then multiply the result by 2. But we've overcounted those points for which both $a \le \sqrt{X}$ and $b \le \sqrt{X}$. So we get

$$\sum_{n\le X} d(n) = 2\sum_{\substack{ab\le X\\ a\le\sqrt{X}}} 1 - \sum_{a,b\le\sqrt{X}} 1.$$

Use this to prove that

$$\sum_{n\le X} d(n) = x\log x + (2\gamma - 1)x + O(\sqrt{x}).$$

(3) (20 points) In this problem, p and q denote primes. Prove the following:

(a) $\displaystyle\sum_{pq\leq x}\frac{1}{pq}=(\log\log x)^2+c\log\log x+O(1)$ for some constant $c>0$.
Hint: try the hyperbola method.

(b) $\displaystyle\sum_{p\leq x}\frac{(\log p)^2}{p}=\frac{(\log x)^2}{2}+O(\log x).$

(c) $\displaystyle\sum_{pq\leq x}\frac{\log p\log q}{pq}=\frac{(\log x)^2}{2}+O(\log x).$

(4) (15 points) Suppose that $\{a(n)\}$ is a nonincreasing sequence of nonnegative real numbers.

(a) Prove that

$$\sum_{n=1}^{\infty}a(n)\leq\sum_{m=0}^{\infty}2^m a(2^m)\leq 2\sum_{n=1}^{\infty}a(n),$$

where all three are infinite if $\sum a(n)$ diverges.

(b) Prove that

$$\sum_{p}a(p)\text{ converges}\qquad\Longleftrightarrow\qquad\sum_{n=1}^{\infty}\frac{a(n)}{\log n}\text{ converges.}$$

(5) (20 points) Let $\chi\neq\chi_0$ be a Dirichlet character mod q, and let $f(n)=\displaystyle\sum_{d|n}\chi(d)$.

(a) Show that $\displaystyle\sum_{n\leq x}\frac{f(n)}{\sqrt{n}}=2L(1,\chi)\sqrt{x}+O(1).$

Hint: try the hyperbola method.

(b) Fact: $f(n)$ is multiplicative, so $f(n)=\prod_{p^k||n}f(p^k)$.
Suppose that χ is a real character. Prove that

$$f(n)\geq 0\quad\text{for all } n\text{, and}\qquad f(n)\geq 1\quad\text{for } n=m^2.$$

(c) Prove, under the same assumption as in (b), that $L(1,\chi)\neq 0$.

(6) (15 points) The Kloosterman sum is defined as

$$K(m,n;q) := \sum_{\substack{d \bmod q \\ (d,q)=1}} e\left(\frac{m\bar{d}+nd}{q}\right),$$

where $\bar{d}$ satisfies $d\bar{d} \equiv 1 \pmod q$. Prove the following:

(a) $K(m,n;q) = K(n,m;q)$.

(b) If $(n,q)=1$ then $K(m,n;q) = K(mn,1;q)$.

(c) Given integers n, q_1, q_2 such that $(q_1,q_2)=1$, show that there exist integers n_1 and n_2 such that

$$n \equiv n_1 q_2^2 + n_2 q_1^2 \pmod{q_1 q_2},$$

and that for these integers we have

$$K(m,n;q_1q_2) = K(m,n_1;q_1)K(m,n_2;q_2).$$

(7) (15 points)

(a) Suppose that $f(n)$ is a multiplicative function (i.e. $f(mn)=f(m)f(n)$ whenever $\gcd(m,n)=1$) such that

$$\lim_{p^m\to\infty} f(p^m) = 0.$$

That is, for every $\varepsilon > 0$ there is an $N(\varepsilon)$ such that $|f(p^m)| < \varepsilon$ whenever $p^m > N(\varepsilon)$. Prove that $f(n) \to 0$ as $n \to \infty$.

(b) Fact: $d(n)$ (see (2)) is multiplicative. Prove that $d(n) \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$.