MATH 205A: ANALYTIC NUMBER THEORY
HOMEWORK 1

This assignment is worth 100 points. You do not need to attempt every problem to earn all of the points. Submissions written in LaTeX will receive 5 bonus points.

(1) (10 points) You may assume that $\pi(x) \sim x/\log x$.
   (a) Let $0 < a < b$. Prove that there exists an x_0 such that for every $x \geq x_0$, there is a prime number between ax and bx.
   (b) Given $n \geq 1$ show that there exists a $k \geq 1$ and a prime p such that
$$10^k n < p < 10^k(n+1).$$
   (c) Given integers $a_1, \ldots, a_m$ with $0 \leq a_i \leq 9$, show that there is a prime number whose decimal expansion has $a_1 \ldots a_m$ as its first m digits.

(2) (25 points)
   (a) Prove that there exist constants c_n (independent of x) such that for every $n \geq 1$,
$$\mathrm{Li}(x) = \frac{x}{\log x}\left(1 + \sum_{k=1}^{n-1} \frac{k!}{\log^k x}\right) + n! \int_2^x \frac{dt}{\log^{n+1} t} + c_n.$$
   (b) Prove that for every $n \geq 1$,
$$\int_2^x \frac{dt}{\log^n t} \ll \frac{x}{\log^n x}.$$
   (c) For $x \geq e$ let
$$I(x) = \int_e^x \log\log t \, dt.$$
   Evaluate $I(x)$ to within an error of at most $O(x/\log^2 x)$.

(3) (25 points) You may assume that $\pi(x) \sim x/\log x$.
   (a) Prove that $\lim\limits_{n\to\infty} \dfrac{p_{n+1}}{p_n} = 1$.
   (b) Suppose that $\{C_n\}$ is a sequence of real numbers satisfying
      (i) $C_{m+n} \leq C_m + C_n$ for all $m, n \in \mathbb{N}$.
      (ii) $C_n \ll \sqrt{n}$.
      (iii) $C_{q+1} \leq 2\sqrt{q}$ for every prime power $q = p^k$.
   Prove that $\limsup\limits_{n\to\infty} \dfrac{C_n}{2\sqrt{n}} \leq 1$.

(4) (25 points) You may assume that $\pi(x) \sim x/\log x$.

(a) Prove that

$$\sum_{n \le x} \frac{p_{n+1} - p_n}{\log n} \sim x.$$

(b) Prove that

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} \le 1 \le \limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n}.$$

(c) Fix $\alpha > 0$. Show that there is a subsequence of primes $\{p_{n_k}\}$ such that $p_{n_k} \sim \alpha k$.

(d) Prove that the set of rationals of the form $\frac{p}{q}$ with p, q prime, is dense in $[0, \infty)$.

(5) (10 points) For each $\alpha > 0$, investigate the convergence of the series

$$\sum_p \frac{1}{p(\log p)^\alpha}.$$

(6) (20 points) Let $\omega(n)$ denote the number of distinct primes dividing n, and let $\Omega(n)$ denote the number of primes dividing n, counted with multiplicity. For example,

$$\omega(2^3 3^2 5^7) = 3, \qquad \Omega(2^3 3^2 5^7) = 3 + 2 + 7 = 12.$$

Prove that there exist constants A, B for which

$$\sum_{n \le x} \omega(n) = x \log\log x + Ax + O\left(\frac{x}{\log x}\right),$$

$$\sum_{n \le x} \Omega(n) = x \log\log x + Bx + O\left(\frac{x}{\log x}\right).$$

Hint: you can use Mertens' second theorem.

(7) (10 points) Show that the statement

$$\sum_{p \le x} \frac{1}{p} = \log\log x + m + o\left(\frac{1}{\log x}\right),$$

where m is a constant, implies the prime number theorem.