CHAPTER 6 PRACTICE QUIZ SOLUTIONS

(1) Compute the integral

$$\int_{|z|=\pi} \frac{z}{\cos z - 1}\, dz.$$

The denominator has a zero of order 2 at 0 since $\cos z - 1 = -z^2/2 + \dots$. So the integrand has a pole of order one at $z = 0$. Since the next largest zeros of $\cos z - 1$ are at $z = \pm 2\pi$, which are both outside the contour, there is only one pole to consider. Thus we have

$$\int_{|z|=\pi} \frac{z}{\cos z - 1}\, dz = 2\pi i \operatorname{Res}(0).$$

To compute the residue, we find that

$$\operatorname{Res}(0) = \lim_{z\to 0} z \cdot \frac{z}{\cos z - 1} = \lim_{z\to 0} \frac{z^2}{-z^2/2 + z^4/4! - \dots} = \lim_{z\to 0} \frac{1}{-1/2 + z^2/4! - \dots} = -2.$$

It follows that the integral equals $-4\pi i$.

(2) With the aid of residues, verify the integral formula

$$\int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4}\, dx = \frac{\pi}{4}.$$

Proof. Since the integrand is even, we will instead show that

$$\int_{-\infty}^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4}\, dx = \frac{\pi}{2}.$$

The zeros of the denominator are at the points where $z^4 + 5z^2 + 4 = 0$, i.e.

$$0 = (z^2 + 4)(z^2 + 1) = (z - 2i)(z + 2i)(z - i)(z + i).$$

In the upper half-plane, the relevant zeros are i and $2i$. Let γ_1 be the interval $[-R, R]$ on the real line and let γ_2 denote the semicircle in the upper half-plane of radius R centered at 0, positively oriented. Then

$$\int_{\gamma_1+\gamma_2} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz = 2\pi i(\operatorname{Res}(i) + \operatorname{Res}(2i)) \tag{0.1}$$

Using the lemma about computing residues of functions of the form P/Q where $P(z_0) \neq 0$ and Q has a simple zero at z_0, we find that

$$\operatorname{Res}(i) = \frac{2i^2 - 1}{4i^3 + 10i} = \frac{-3}{6i} = \frac{-1}{2i}$$

and

$$\operatorname{Res}(2i) = \frac{8i^2 - 1}{32i^3 + 20i} = \frac{-9}{-12i} = \frac{3}{4i}.$$

So the right-hand side of (0.1) is $2\pi i(1/(4i)) = \pi/2$.

By a lemma from the book, we have $\lim_{R\to\infty} \int_{\gamma_2} = 0$ since the degree of the denominator is 4 and the degree of the numerator is 2. The result follows. □

(3) Use Rouché's Theorem to prove that the polynomial $P(z) = z^5 + 3z^2 + 1$ has exactly three zeros in the annulus $1 < |z| < 2$. *Hint:* prove that it has five zeros in $|z| < 2$ and two zeros in $|z| < 1$.

Proof. Apply Rouché's Theorem to the functions $f(z) = z^5$ and $g(z) = z^5 + 3z^2 + 1$ on $|z| = 2$. We have

$$|f(z) - g(z)| = |3z^2 + 1| \leq 3|z|^2 + 1 = 3 \cdot 2^2 + 1 = 13 < 32 = |z|^5 = |f(z)|.$$

So g has 5 zeros inside $|z| < 2$.

Now apply Rouché's Theorem to the functions $f(z) = 3z^2$ and $g(z) = z^5 + 1$ on $|z| = 1$. We have

$$|f(z) - g(z)| = |z^5 + 1| \leq |z|^5 + 1 = 2 < 3 = 3|z|^2 = |f(z)|.$$

So g has 2 zeros inside $|z| < 1$.

What happens on $|z| = 1$? We have $|P(z)| = |z^5 + 3z^2 + 1| \geq 3|z|^2 - |z|^5 - 1 = 1 > 0$ so P has no zeros on $|z| = 1$.

We conclude that g has 3 zeros in the annulus. $\square$