CHEAT SHEET

Definition 1. If $z = x + iy$ then e^z is defined to be the complex number $e^x(\cos y + i \sin y)$.

Proposition 2 (De Moivre's Formula). *If $n = 1, 2, 3, \ldots$ then*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta.$$

Definition 3. A *domain* is an open connected set.

Definition 4. A complex-valued function f is said to be *analytic* on an open set D if it has a derivative at every point of D.

Theorem 5. *If $f(z) = u(x, y) + iv(x, y)$ is analytic in D then the* Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

must hold at every point in D.

Theorem 6 (The Fundamental Theorem of Algebra). *Every nonconstant polynomial with complex coefficients has at least one zero in $\mathbb{C}$.*

Theorem 7. *Given any complex number z, we define*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Definition 8. The *principal value of the logarithm* is defined as

$$\operatorname{Log} z := \operatorname{Log}|z| + i \operatorname{Arg} z.$$

Theorem 9 (The ML-inequality). *If f is continuous on the contour Γ and if $|f(z)| \leq M$ for all z on Γ then*

$$\left| \int_\Gamma f(z)\, dz \right| \leq M \cdot \text{length}(\Gamma).$$

Theorem 10 (Independence of Path). *Suppose that the function $f(z)$ is continuous in a domain D and has an antiderivative F throughout D. Then for any contour Γ in D with initial point z_I and terminal point z_T we have*

$$\int_\Gamma f(z)\, dz = F(z_T) - F(z_I).$$

Definition 11. A domain D is *simply connected* if one of the following (equivalent) conditions holds:

- every loop in D can be continuously deformed in D to a point.
- if Γ is any simple closed contour in D then $\text{int}(\Gamma) \subseteq D$.

Theorem 12 (Cauchy's Integral Theorem). *If f is analytic in a simply connected domain D then*

$$\int_\Gamma f = 0 \qquad \textit{for all loops } \Gamma \subseteq D.$$

Theorem 13 (Morera's Theorem). *If f is continuous in a domain D and if*

$$\int_\Gamma f = 0 \qquad \textit{for all loops } \Gamma \subseteq D$$

then f is analytic in D.

Theorem 14 (Cauchy's (generalized) Integral Formula)**.** *If f is analytic inside and on the simple positively oriented loop Γ and if z is any point inside Γ then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

for $n = 1, 2, 3, \ldots$.

Theorem 15 (Liouville's Theorem)**.** *The only bounded entire functions are the constant functions.*

Theorem 16 (The Maximum Modulus Principle)**.** *A function analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary.*

Theorem 17 (The Taylor Series Theorem)**.** *If f is analytic in the disk $|z - z_0| < R$ then the Taylor series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

converges to $f(z)$ for all z in the disk. Furthermore, the convergence is uniform in any closed subdisk $|z - z_0| \le R' < R$.

Definition 18. A point z_0 is called a *zero of order m* for f if f is analytic at z_0 and if $f^{(n)}(z_0) = 0$ for $0 \le n \le m - 1$ but $f^{(m)}(z_0) \neq 0$.

Definition 19. If f has an isolated singularity at z_0 and if

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is its Laurent expansion around z_0 then

(1) If $a_n = 0$ for all $n < 0$ then f has a *removable singularity* at z_0.
(2) If $a_{-m} \neq 0$ for some $m > 0$ but $a_n = 0$ for all $n < -m$ then f has a *pole of order m* at z_0.
(3) If $a_n \neq 0$ for infinitely many negative n then f has an *essential singularity* at z_0.

Theorem 20 (Picard's Theorem)**.** *A function with an essential singularity assumes every complex number, with possibly one exception, as a value in any neighborhood of this singularity.*

Definition 21. If f has an isolated singularity at z_0 then the coefficient a_{-1} of $1/(z - z_0)$ in the Laurent expansion of f around z_0 is called the *residue* of f at z_0.

Theorem 22 (The Residue Theorem)**.** *If Γ is a simple positively oriented loop and f is analytic inside and on Γ except at the points $z_1, \ldots, z_n$ inside Γ then*

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz = \sum_{k=1}^{n} \operatorname{Res}(f; z_k).$$

Theorem 23 (The Argument Principle)**.** *If f is analytic and nonzero at each point of a simple positively oriented loop Γ and is meromorphic inside Γ then*

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)}\, dz = Z - P,$$

where Z is the number of zeros of f inside Γ and P is the number of poles inside Γ, counted with multiplicity.

Theorem 24 (Rouché's Theorem)**.** *If f and g are analytic inside and on a simple loop Γ and if*

$$|f(z) - g(z)| < |f(z)| \qquad \text{for all } z \text{ on } \Gamma$$

then f and g have the same number of zeros (counting multiplicities) inside Γ.

Final Exam Information

The final exam will be worth 40 total points, broken up in the following way:

15 points: Chapter 6 Quiz (3 questions)

15 points: Cumulative Quiz (3 questions very similar to exercises from HW 1–4)

10 points: Definitions / Theorems (See Cheat Sheet above)

To help you practice for the Chapter 6 Quiz portion, here are some exercises to do from Sec. 6.7:

$$(6.7) \quad 1, 2, 3, 6, 7, 8$$

Practice Chapter 6 Quiz:

(1) Compute the integral
$$\int_{|z|=\pi} \frac{z}{\cos z - 1}\, dz.$$

(2) With the aid of residues, verify the integral formula
$$\int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4}\, dx = \frac{\pi}{4}.$$

(3) Use Rouché's Theorem to prove that the polynomial $P(z) = z^5 + 3z^2 + 1$ has exactly three zeros in the annulus $1 < |z| < 2$. *Hint:* prove that it has five zeros in $|z| < 2$ and two zeros in $|z| < 1$.