Math 132 Quiz 4 Solutions

1. (5 points) Does the function

$$f(z) = \frac{z^5}{(6\sin z - 6z + z^3)^2}$$

have a zero or a pole at $z = 0$? What is the order of the zero or pole? *Hint:* work with $1/f(z)$.

We start by writing out the Taylor expansion of the denominator. We have

$$(6\sin z - 6z + z^3)^2 = \left(6z - 6z^3/6 + 6z^5/120 - \ldots - 6z + z^3\right)^2 = \left(z^5/20 - \ldots\right)^2 = z^{10}/400 + \ldots.$$

So the denominator has a zero of order 10 at $z = 0$. Since the numerator has a zero of order 5 at $z = 0$, we conclude that $f(z)$ has a pole of order 5 at $z = 0$.

2. (5 points) Prove or disprove: if $f(z)$ has a zero of order 6 at $z = 0$ and $g(z)$ has a pole of order 3 at $z = 0$ then the function $h(z) = f(z)g(z^2)$ can be redefined at $z = 0$ so that it is analytic at $z = 0$.

We will show that $h(z)$ has a removable singularity at $z = 0$, since that is equivalent to being able to redefine h so that it's analytic at $z = 0$.

Since f has a zero of order 6 we have $f(z) = z^6 f_1(z)$ where $f_1(z)$ is analytic and nonzero at 0, and since $g(z)$ has a pole of order 3 we have $g(z) = z^{-3} g_1(z)$ where g_1 is analytic and nonzero at 0. Thus, for $z \neq 0$ we have

$$h(z) = f(z)g(z^2) = z^6 f_1(z) z^{-6} g_1(z^2) = f_1(z) g_1(z^2).$$

The Laurent expansion of the latter function will have no negative powers of z since f_1 and g_1 are both analytic at $z = 0$. Thus h has a removable singularity at $z = 0$.

3. (5 points) Find the first two nonzero terms in the Laurent expansion of the function

$$\frac{1}{\operatorname{Log}(1-z)} + \frac{1}{z}$$

in the region $0 < |z| < 1$.

We start with $\operatorname{Log}(1-z) = -z - z^2/2 - z^3/3 - \ldots$ from which we obtain

$$\begin{aligned}\frac{1}{\operatorname{Log}(1-z)} &= \frac{1}{-z - z^2/2 - z^3/3 - \ldots} = \frac{-1}{z} \cdot \frac{1}{1 - (-z/2 - z^2/3 - z^3/4 - \ldots)} \\ &= \frac{-1}{z} \sum_{n=0}^{\infty} (-z/2 - z^2/3 - z^3/4 - \ldots)^n = \frac{-1}{z}\left(1 - \frac{z}{2} - \frac{z^2}{12} - \ldots\right).\end{aligned}$$

It follows that

$$\frac{1}{\operatorname{Log}(1-z)} + \frac{1}{z} = \frac{1}{2} + \frac{z}{12} + \ldots.$$