MATH 132 QUIZ 4 (PRACTICE)

(1) Compute the Taylor series for $g(z) = \dfrac{1}{(1-z)^3}$ around $z = 0$ and find the radius of convergence R.

True or false: this series converges at every point on the boundary $|z| = R$.

First we notice that $\dfrac{1}{(1-z)^3} = \frac{1}{2}f''(z)$, where $f(z) = \dfrac{1}{1-z}$. So

$$\frac{1}{(1-z)^3} = \frac{1}{2}\frac{d^2}{dx^2}\sum_{n=0}^{\infty} z^n = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)z^{n-2} = 1 + 3z + 6z^2 + 10z^3 + 15z^4 + \dots.$$

Since the radius of convergence of the Taylor series for f around $z = 0$ is $R = 1$, the radius of convergence of the Taylor series for g around $z = 0$ is also $R = 1$. The series doesn't converge for any z on the boundary $|z| = 1$ (but you only need to show this for a single value of z to conclude that the statement above is false). For example, at $z = 1$ we have the series $1 + 3 + 6 + 10 + 15 + \dots$ which is obviously divergent.

(2) Compute the Laurent series of $f(z) = \dfrac{1}{z^3(1+z)}$ in the regions

(a) $0 < |z| < 1$
(b) $|z + 1| > 1$
Hint: for part (b) you'll want to use your answer from #1 above.

(a) In $0 < |z| < 1$ we have

$$f(z) = \frac{1}{z^3} \cdot \frac{1}{1-(-z)} = \frac{1}{z^3}\sum_{n=0}^{\infty}(-z)^n = \sum_{n=0}^{\infty}(-1)^n z^{n-3} = z^{-3} - z^{-2} + z^{-1} - 1 + z - \dots$$

(b) In $|z + 1| > 1$ we have

$$f(z) = \frac{1}{1+z} \cdot \frac{1}{((1+z)-1)^3} = \frac{1}{(1+z)^4} \cdot \frac{1}{(1-(1+z)^{-1})^3}$$
$$= \frac{1}{(1+z)^4}\sum_{n=2}^{\infty} n(n-1)(1+z)^{-n+2} = (1+z)^{-4} + 3(1+z)^{-5} + 6(1+z)^{-6} + 10(1+z)^{-7} + \dots$$

(3) Classify the zero or singularity of the function

$$h(z) = \frac{\mathrm{Log}^2(z)}{(z-1)e^z}$$

at the point $z = 1$. If the singularity is a zero or pole, state its order.

The function e^z is analytic and nonzero at $z = 1$, so it will not contribute to a zero or singularity of $h(z)$ there. (It will affect the coefficients of the Laurent series, just not the lowest exponent). Thus it is safe (for this problem) to ignore the e^z factor.

The factor $(z - 1)$ is already in the right form for a Laurent series centered at $z = 1$, so let's focus first on the log term. Near $z = 1$ we have

$$\mathrm{Log}(z) = \mathrm{Log}(1-(1-z)) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n} = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \dots.$$

It follows that

$$\mathrm{Log}^2(z) = \left((z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \dots\right)^2 = (z-1)^2 - (z-1)^3 + \frac{11}{12}(z-1)^4 - \dots$$

and therefore

$$\frac{\mathrm{Log}^2(z)}{z-1} = (z-1) - (z-1)^2 + \dots$$

which has a simple zero (a zero of order one) at $z = 1$.