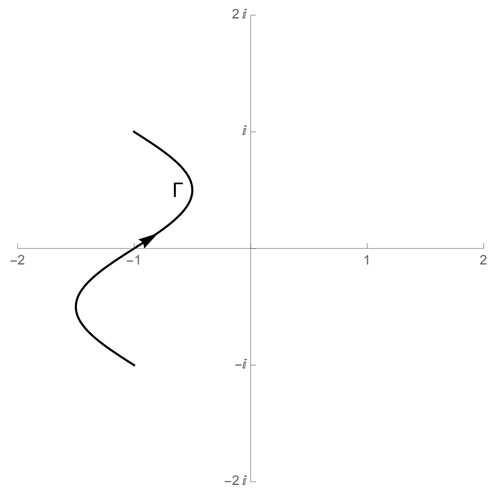
Math 132 Quiz 3 Solutions

1. (5 points) Compute $\displaystyle\int_\Gamma \frac{dz}{z}$, where Γ is the contour shown below connecting $-1-i$ to $-1+i$.



See practice quiz 3 for the solution.

2. (5 points) Suppose f is entire and $|f(z)| \le A + B|z|^{3/2}$. Prove that f is a linear polynomial.

Hint: Use Cauchy's integral formula to prove that $f''(z) = 0$ for all $z \in \mathbb{C}$.

Proof. Let $z \in \mathbb{C}$ and let $R > 0$ be large enough so that $R > |z|$ (eventually we will take $R \to \infty$). By Cauchy's Integral Formula and the ML inequality we have

$$|f''(z)| \le \frac{1}{\pi}\left|\int_{|\zeta|=R} \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta\right| \le \frac{1}{\pi}\frac{A + BR^{3/2}}{(R - |z|)^3} \cdot 2\pi R.$$

In the denominator we used the reverse triangle inequality $|\zeta - z| \ge |\zeta| - |z| = R - z$. The total degree (in R) of the numerator is $5/2$ while the degree of the denominator is 3, so the right-hand side goes to zero as $R \to \infty$. It follows that $f''(z) = 0$ for all z, and thus f is a linear polynomial. $\square$

3. (5 points) Evaluate

$$\int_\Gamma \frac{\sin \pi z}{(z^2 - 1)^2}\, dz,$$

where Γ is the circle $|z| = 2$ traversed once counterclockwise.

The circle $|z| = 2$ contains both poles of the integrand (± 1) in its interior. So we deform Γ into two small circles $\gamma_1 + \gamma_2$ centered around -1 and 1, respectively. Then

$$\int_\Gamma \frac{\sin \pi z}{(z^2 - 1)^2}\, dz = \int_{\gamma_1} \frac{\sin \pi z/(z-1)^2}{(z+1)^2}\, dz + \int_{\gamma_2} \frac{\sin \pi z/(z+1)^2}{(z-1)^2}\, dz.$$

I have written the integrals so that the numerator is an analytic function inside and on γ_j for $j = 1, 2$. So we can apply Cauchy's integral formula directly to get

$$\int_\Gamma \frac{\sin \pi z}{(z^2 - 1)^2}\, dz = \frac{2\pi i}{1!} f'(-1) + \frac{2\pi i}{1!} g'(1),$$

where $f(z) = \sin \pi z/(z - 1)^2$ and $g(z) = \sin \pi z/(z + 1)^2$. We compute that $f'(-1) = g'(1) = -\pi/4$, from which it follows that

$$\int_\Gamma \frac{\sin \pi z}{(z^2 - 1)^2}\, dz = -\pi^2 i.$$