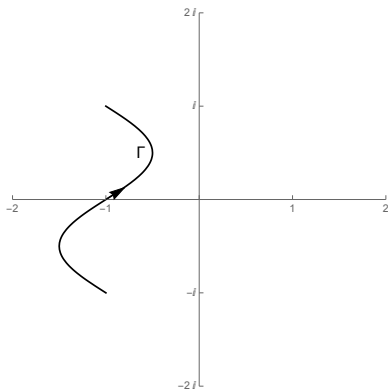


MATH 132 QUIZ 3 (PRACTICE) SOLUTIONS

- (1) Let Γ be the contour shown below from $-1 - i$ to $-1 + i$.



Compute $\int_{\Gamma} \frac{dz}{z}$.

We need an antiderivative for $1/z$ which is analytic in an open neighborhood of the contour Γ . There are many reasonable choices, but I'll pick $F(z) = \text{Log}_0(z)$ (this is analytic in $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$). Then the integral equals

$$\begin{aligned} F(-1 + i) - F(-1 - i) &= \text{Log}_0(-1 + i) - \text{Log}_0(-1 - i) \\ &= \text{Log} \sqrt{2} + i \text{Arg}_{(0, 2\pi)}(-1 + i) - \text{Log} \sqrt{2} - i \text{Arg}_{(0, 2\pi]}(-1 - i) = 3\pi i/4 - 5\pi i/4 = -\pi i/2. \end{aligned}$$

- (2) Let C be the positively oriented circle $|z| = 2$. Compute $\int_C \frac{e^{2z}}{(1 - z)^3} dz$.

By Cauchy's Integral Formula, this equals $-\frac{2\pi i}{2!} f''(1)$ (you need to factor out -1 from the denominator to put it in the right form), where $f(z) = e^{2z}$. Since $f''(z) = 4e^{2z}$ the integral equals $-4\pi i e^2$.

- (3) Suppose that $P(z)$ is a polynomial with a zero of order m at $z = 0$ and no other zeros inside the closed unit disk $|z| \leq 1$. Prove that

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{P'(z)}{P(z)} dz = m.$$

Hint: first find a way to simplify $P'(z)/P(z)$.

Writing $P(z) = z^m Q(z)$ where $Q(z) \neq 0$ for all $|z| \leq 1$, we find that

$$\frac{P'(z)}{P(z)} = \frac{mz^m Q(z) + z^m Q'(z)}{z^m Q(z)} = \frac{m}{z} + \frac{Q'(z)}{Q(z)}.$$

The function Q'/Q is analytic in $|z| \leq 1$ since $Q(z) \neq 0$ there. So its loop integrals are zero by Cauchy's integral theorem. Thus

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{P'(z)}{P(z)} dz = \frac{1}{2\pi i} \int_{|z|=1} \frac{m}{z} dz = m.$$

- (4) Suppose that f is an entire function with $\operatorname{Re} f(z) \leq M$ (for some positive constant M) for all $z \in \mathbb{C}$. Prove that f is constant.

Hint: First prove that $e^{f(z)}$ is constant. Then conclude that $f'(z) = 0$ and thus $f(z)$ is constant.

Let $g(z) = e^{f(z)}$. Then $|g(z)| = e^{\operatorname{Re} f(z)} \leq e^M$. Since g is entire and bounded, it must be a constant. Therefore

$$0 = g'(z) = f'(z)e^{f(z)}.$$

It follows that $f'(z) = 0$ and thus f is a constant.