Math 132 Quiz 2

1. (5 points)

   (a) What is the inverse image of the positive real line

   $$\mathbb{R}_{>0} = \{z \in \mathbb{C} : \operatorname{Im} z = 0, \operatorname{Re} z > 0\}$$

   under the map $z \mapsto \dfrac{1}{z+i}$? That is, what subset of $\mathbb{C}$ maps to $\mathbb{R}_{>0}$ under $z \mapsto \dfrac{1}{z+i}$?

   I would prefer to draw this out graphically, but that's hard to do by typing so I'll do it another way. Suppose $\dfrac{1}{z+i} = x$ where x is a positive real number. Then $z = 1/x - i$. The set of all such numbers is the set $\{z \in \mathbb{C} : \operatorname{Im} z = -1, \operatorname{Re} z > 0\}$.

   (b) Find a branch of the multi-valued function $\log\left(\dfrac{1}{z+i}\right)$ that is analytic in the upper half-plane $\{\operatorname{Im}(z) > 0\}$.

   The branch cut of the function $\operatorname{Log}_0(z)$ is $\mathbb{R}_{\geq 0}$, so by (a), $\operatorname{Log}_0\left(\dfrac{1}{z+i}\right)$ is analytic in $\mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = -1, \operatorname{Re} z \geq 0\}$. Since this set contains the upper half-plane $\{\operatorname{Im} z > 0\}$ we are done.

   (c) Compute the value of your function in (b) at the point $z = i$.

   $\operatorname{Log}_0(\frac{1}{2i}) = \operatorname{Log}(\frac{1}{2}) + i \operatorname{Arg}_{(0,2\pi]}(\frac{1}{2i}) = -\operatorname{Log} 2 + \pi i/2.$

2. (5 points) Show that if the rational function $R(z)$ has a pole of order m at $z = 0$ then its derivative has a pole of order $m+1$ at $z = 0$.

   Write $R(z) = \dfrac{p(z)}{z^m q(z)}$ where p, q are polynomials and $p(0) \neq 0$ and $q(0) \neq 0$. Then

   $$R'(z) = -mz^{-m-1}\frac{p(z)}{q(z)} + z^{-m}\left(\frac{p(z)}{q(z)}\right)' = \frac{-mp(z)/q(z) + z(p(z)/q(z))'}{z^{m+1}}.$$

   This looks like it has a pole of order $m+1$ at $z = 0$, but we're not done yet! We need to make sure that the numerator isn't zero at $z = 0$ (if it is, then we'll have cancellation and the pole order will be smaller). At $z = 0$ the numerator equals $-mp(0)/q(0) \neq 0$ by assumption. That finishes the proof.

3. (5 points) Consider the multi-valued function $(z^4 - 1)^{1/4}$. Verify that the function

   $$g(z) = e^{\pi i/4} e^{\frac{1}{4}\operatorname{Log}(1-z^4)}$$

   is a branch of $(z^4 - 1)^{1/4}$ that is analytic in the interior of the unit disk $|z| < 1$.

   Note: you must show two things: that $[g(z)]^4 = z^4 - 1$, and that g is analytic in $|z| < 1$.

Let's first check that $[g(z)]^4 = z^4 - 1$. We have

$$g(z)^4 = e^{\pi i} e^{\mathrm{Log}(1-z^4)} = e^{\pi i}(1 - z^4) = z^4 - 1.$$

Now we need to show that $g(z)$ is analytic in $|z| < 1$. Again, normally I'd do this graphically but I'll do it a different way here so I can type it easier.

Suppose that $1 - z^4$ is on the branch cut of Log, i.e. that $1 - z^4 = x$ where $x < 0$. Then $z^4 = 1 - x > 1$. Therefore $|z| > 1$. So in the open unit disk $|z| < 1$ the function $\mathrm{Log}(1 - z^4)$ is analytic since $1 - z^4$ avoids the negative real line when $|z| < 1$. Thus g is analytic in $|z| < 1$.