MATH 132 QUIZ 2 (PRACTICE) SOLUTIONS

(1) (a) Compute all values of $(-1)^i$
We have $(-1)^i = e^{i\log(-1)} = e^{i(i\pi+2\pi ik)} = e^{-\pi-2\pi k}$ for $k \in \mathbb{Z}$.

(b) Find a branch of the multivalued function z^i which is continuous for all z in the left half-plane $\mathrm{Re}(z) < 0$.
The function $f(z) = e^{i\,\mathrm{Log}_0(z)}$ is analytic in $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, so this choice of branch works.

(c) Repeat part (b), but with the additional requirement that, with your choice of branch, $(-1)^i = e^{5\pi}$.
Let's still pick a branch of log that is analytic in $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, but shift things so that the value at -1 is correct. This leaves us with $e^{i\,\mathrm{Log}_{2\pi k}(z)}$ for some $k \in \mathbb{Z}$. The choice $k = -3$ works because $e^{i\,\mathrm{Log}_{-6\pi}(-1)} = e^{i(-5\pi i)} = e^{5\pi}$. So one correct answer would be $g(z) = e^{i\,\mathrm{Log}_{-6\pi}(z)}$ (but there are many other correct answers).

(2) Prove that if the polynomial $p(z)$ has a zero of order m at z_0 then $p'(z)$ has a zero of order $m-1$ at z_0.

Proof. Suppose $p(z)$ has a zero of order m at z_0. Then $p(z) = (z-z_0)^m q(z)$, where $q(z_0) \neq 0$. Taking the derivative, we find that $p'(z) = m(z-z_0)^{m-1}q(z) + (z-z_0)^m q'(z) = (z-z_0)^{m-1}\big(mq(z) + (z-z_0)q'(z)\big)$. This function has a zero at z_0 of order $\geq m-1$. To show that the order is exactly $m-1$, we need to show that $mq(z)+(z-z_0)q'(z)\big|_{z=z_0} \neq 0$. But this is true because $mq(z) + (z-z_0)q'(z)\big|_{z=z_0} = mq(z_0) \neq 0$. $\square$

(3) (a) What are all of the possible values of $\log(i-z)$ at the point $z = 1$?
$\log(i-1) = \mathrm{Log}\,|i-1| + i\arg(i-1) = \mathrm{Log}\sqrt{2} + i(3\pi/4 + 2\pi k)$, $k \in \mathbb{Z}$.

(b) Where is the principal branch $\mathrm{Log}(i-z)$ analytic?
The principal branch $\mathrm{Log}\,z$ is analytic in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, so we need to figure out the inverse image of the set $\mathbb{R}_{\leq 0}$ under the map $z \mapsto i-z$. You can do this geometrically (draw a picture) or algebraically. For the sake of being able to type it up easily, I'll do it algebraically here (although I personally prefer a geometric argument). If $z = x+iy$ then $i-z = i-(x+iy) = -x + i(1-y)$. This is in $\mathbb{R}_{\leq 0}$ if $1-y = 0$ and $-x \leq 0$, i.e. if $y = 1$ and $x \geq 0$. So the function $\mathrm{Log}(i-z)$ is analytic in $\mathbb{C} \setminus \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) = 1\}$.

(c) Find a branch of $\log(i-z)$ that takes the value $\frac{1}{2}\mathrm{Log}\,2 - \frac{5\pi i}{4}$ at $z = 1$.
This value corresponds to $k = -1$ in part (a), and one branch of log that makes this work is $\mathrm{Log}_{-2\pi}(i-z)$.