Math 132 Quiz 1 Solutions

1. (5 points)

   (a) Prove that the only real-valued entire functions are the constant functions.

   Proof. Suppose that $f(z) = u(x, y) + iv(x, y)$ is a real-valued entire function. Then $v(x, y) = 0$. By the Cauchy-Riemann equations it follows that $u_x = v_y = 0$. So $f'(z) = u_x + iv_x = 0$. As we proved in class, if f is analytic and $f' = 0$ then f is a constant. $\square$

   (b) Suppose that f and $\bar{f}$ are entire. Prove that f is a constant function.

   Proof. If f and $\bar{f}$ are entire then so are the functions $\frac{1}{2}(f + \bar{f}) = \mathrm{Re}(f)$ and $\frac{1}{2i}(f - \bar{f}) = \mathrm{Im}(f)$. Both of these functions are real-valued, so by part (a) they are both constants. It follows that f is a constant function. $\square$

2. (5 points)

   (a) Next week we will learn that the logarithm of a complex number is defined as

   $$\mathrm{Log}(z) := \log|z| + i\,\mathrm{Arg}(z),$$

   where log is the usual natural logarithm defined on positive real numbers. Compute:

   $$\begin{aligned} \mathrm{Log}(1) &= \boxed{0} \\ \mathrm{Log}(i) &= \boxed{i\pi/2} \\ \mathrm{Log}(-1) &= \boxed{i\pi} \\ \mathrm{Log}(-i) &= \boxed{-i\pi/2} \end{aligned}$$

   (b) Now we can define complex exponentiation. If $a, b \in \mathbb{C}$ then we define

   $$a^b := e^{b\,\mathrm{Log}(a)}.$$

   Compute i^i. $\boxed{i^i = e^{i\,\mathrm{Log}\,i} = e^{i(i\pi/2)} = e^{-\pi/2}}$

3. (5 points) Prove that if $|z| = 1$ and $z \neq -1$ then

   $$\mathrm{Arg}\left(\frac{z-1}{z+1}\right) = \begin{cases} \frac{\pi}{2} & \text{if } \mathrm{Im}(z) > 0, \\ -\frac{\pi}{2} & \text{if } \mathrm{Im}(z) < 0. \end{cases}$$

Proof. Start with $\dfrac{z-1}{z+1} = \dfrac{z-1}{z+1} \cdot \dfrac{z^{-1/2}}{z^{-1/2}} = \dfrac{z^{1/2} - z^{-1/2}}{z^{1/2} + z^{-1/2}}$. Since $|z| = 1$ we can write $z = e^{i\theta}$, from which it follows that

$$\frac{z-1}{z+1} = \frac{e^{i\theta/2} - e^{-i\theta/2}}{e^{i\theta/2} + e^{-i\theta/2}} = i\frac{\sin(\theta/2)}{\cos(\theta/2)} = i\tan(\theta/2).$$

Note that $\tan(\theta/2) \in \mathbb{R}$ so only its sign ($\pm$) affects the argument of $i\tan(\theta/2)$. If $\mathrm{Im}(z) > 0$ then $\tan(\theta/2) > 0$ and thus $\mathrm{Arg}\left(\frac{z-1}{z+1}\right) = \pi/2$. Similarly, if $\mathrm{Im}(z) < 0$ then $\mathrm{Arg}\left(\frac{z-1}{z+1}\right) = -\pi/2$. $\square$