

MATH 132 QUIZ 1 (PRACTICE) SOLUTIONS

- (1) Find the range of the function $f(z) = e^z$ with domain $\operatorname{Re}(z) < 0$.

Answer: the punctured unit disk $0 < |z| < 1$.

Proof. First, notice that $|e^z| = |e^x e^{iy}| = e^x < 1$ since $x < 0$. So f maps into the open unit disk $|z| < 1$. Is every point in the disk in the range of f ? Since $e^x \neq 0$ the origin $z = 0$ is not in the range. But every other point is: suppose $re^{i\theta}$ is in the punctured open unit disk (so $0 < r < 1$). Then $r = e^x$ for some x (you can take $x = \log r < 0$) and $\theta = y$ for some y (there are infinitely many choices of y such that $e^{iy} = e^{i\theta}$, but the most obvious one is to take $y = \theta$). $\square$

- (2) Suppose that $|\alpha| < 1$. Prove that $\left| \frac{z - \alpha}{1 - \bar{\alpha}z} \right| = 1$ if and only if $|z| = 1$.

Proof. Since $|w| \geq 0$ for all $w \in \mathbb{C}$ we have

$$\left| \frac{z - \alpha}{1 - \bar{\alpha}z} \right| = 1 \iff |z - \alpha| = |1 - \bar{\alpha}z| \iff |z - \alpha|^2 = |1 - \bar{\alpha}z|^2.$$

Using that $|w|^2 = w\bar{w}$ we continue the computation above as follows:

$$\begin{aligned} &\iff z\bar{z} - \alpha\bar{z} - \bar{\alpha}z + \alpha\bar{\alpha} = 1 - \bar{\alpha}z - \alpha\bar{z} - \alpha\bar{\alpha}z\bar{z} \\ &\iff |z|^2 + |\alpha|^2 = 1 + |z|^2|\alpha|^2 \\ &\iff |z|(1 - |\alpha|^2) = 1 - |\alpha|^2 \\ &\iff |z| = 1. \end{aligned}$$

In the last step we used that $|\alpha| < 1$, so $1 - |\alpha| \neq 0$. $\square$

- (3) (a) Find an entire function f whose real part is $u(x, y) = x^2 + xy - y^2$.

We need to find a harmonic conjugate v for u . Starting with $u_x = 2x + y$ we know by the Cauchy-Riemann equations that v should satisfy $v_y = u_x$, so $v = \int (2x + y) dy = 2xy + \frac{1}{2}y^2 + g(x)$ for some function g depending only on x . By the other Cauchy-Riemann equation $u_y = -v_x$ we find that $2y + g'(x) = -x + 2y$ so $g(x) = -\frac{1}{2}x^2 + C$. We can choose any $C \in \mathbb{C}$ so for simplicity, we choose $C = 0$. Since u and v satisfy the CR equations at every $z = x + iy$ (by construction) we know that $f(z) = u(x, y) + iv(x, y) = (x^2 + xy - y^2) + i(\frac{1}{2}y^2 + 2xy - \frac{1}{2}x^2)$ is entire.

- (b) Explain why there is no entire function whose real part is $u(x, y) = y^2 + 2xy$.

Compute that $u_{xx} + u_{yy} = 2 \neq 0$ so u is not harmonic. Since the real/imaginary parts of any analytic function are harmonic, we see that u is not the real part of any analytic function.