MATH 132 HOMEWORK 3

- (3.3) 12, 13, 14
- (3.5) 15, 16
- (4.1) 9, 11
- (4.2) 5, 8, 10, 14
- (4.3) 2, 3, 4
- The Riemann zeta function is defined via the series
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$
where $s = \sigma + it$ is a complex variable (it is customary to use the $s = \sigma + it$ notation instead of the usual $z = x + iy$ because this is what Riemann used originally).
  a) Determine the region of absolute convergence of $\zeta(s)$; that is, give conditions on σ, t which guarantee that the series
  $$\sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right|$$
  converges. (After taking absolute values, you are left with a series of real, positive terms, and you can use what you already know about series to finish the problem.) We will prove later that $\zeta(s)$ is analytic in this region.
  b) Assuming for the moment that differentiation of series works term-by-term, compute $\zeta'(s)$ as an infinite series of the form $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^s}$.
  c) Polynomials are uniquely determined by their roots, up to a constant multiple. It turns out that something similar is also true for entire functions; in particular we have the following formula for $\sin(\pi z)$:
  $$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \pi z \left(1 - \frac{z^2}{1^2}\right)\left(1 - \frac{z^2}{2^2}\right)\left(1 - \frac{z^2}{3^2}\right) \cdots$$
  (Hopefully we'll have time to prove this later.) Using the product formula for $\sin \pi z$, prove that $\zeta(2) = \frac{\pi^2}{6}$. *Hint:* If you multiply out the factors on the right-hand side of the product formula above, what is the coefficient of z^3? What is the coefficient of z^3 in the Maclaurin series for $\sin \pi z$?

Extra Practice (do not turn in):
(3.3) 1, 3, 15 (3.5) 1, 3, 4, 5 (4.1) 1, 5, 10 (4.2) 3, 6, 7, 9, 11, 12, 13 (4.3) 1(a–h), 5