MATH 131A HOMEWORK 9

(1) Using the limit definition of the derivative, prove that the function
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$
is differentiable at $x = 0$. *Hint:* you should first guess what $f'(0)$ is.

(2) Suppose that f is defined on an open interval I containing a. Prove that $f'(a)$ exists if and only if there is a function $\varepsilon : I \to \mathbb{R}$ such that
$$f(x) - f(a) = (x-a)[f'(a) + \varepsilon(x)] \quad \text{and} \quad \lim_{x \to a} \varepsilon(x) = 0.$$
(For the $\Leftarrow$ direction, you don't know that $f'(a)$ exists so this problem is saying: Assume that there is a function $\varepsilon : I \to \mathbb{R}$ and a number m such that $\lim_{x\to a} \varepsilon(x) = 0$ and $f(x) - f(a) = (x-a)[m + \varepsilon(x)]$. Show that $f'(a)$ exists and that it equals m.)

(3) Using the fact that $(\cos x)' = -\sin x$, prove that
$$|\cos x - \cos y| \leq |x - y|$$
for all $x, y \in \mathbb{R}$. *Hint:* use the MVT.

(4) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function. *Hint:* show that $f'(y) = 0$ for all $y \in \mathbb{R}$ by using the squeeze theorem and the definition of the derivative.

(5) Suppose that f and g are differentiable on $\mathbb{R}$ and that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$. Prove that $f(x) \leq g(x)$ for all $x \geq 0$. *Hint:* consider $h = g - f$.

Extra practice (do not turn these in)

(1) Exercises 28.3, 28.7
(2) Exercises 29.1, 29.4, 29.7, 29.10, 29.11, 29.14, 29.17

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this problem won't be graded but i think it's cool

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$ and suppose that
$$s = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1.$$
We will prove that there exists a point $a \in \mathbb{R}$ such that $f(a) = a$ (i.e. f has a fixed point).

(i) Pick any point $a_1 \in \mathbb{R}$. For each $n \geq 1$ define $a_{n+1} = f(a_n)$. So
$$a_2 = f(a_1), \quad a_3 = f(a_2) = f(f(a_1)), \ldots.$$
Prove that (a_n) is a Cauchy sequence. *Hint:* use the MVT to show that
$$|a_{n+1} - a_n| \leq s|a_n - a_{n-1}|,$$
where s is the sup defined above.

(ii) Prove that there exists some $a \in \mathbb{R}$ such that $f(a) = a$. *Hint:* since f is differentiable on $\mathbb{R}$, it is continuous on $\mathbb{R}$.