MATH 131A HOMEWORK 8

(1) (a) Suppose that f is uniformly continuous on a bounded set S. Prove that f is bounded on S (i.e. there exists a number M such that $|f(x)| \leq M$ for all $x \in S$).
Hint: suppose f isn't bounded, then use Bolzano-Weierstrass to construct a Cauchy sequence (a_n) such that $\lim f(a_n) = \pm\infty$, then derive a contradiction.
(b) Use (a) to prove that $f(x) = 1/x^2$ is not uniformly continuous on $(0, 1)$.

(2) A function $f : X \to \mathbb{R}$ is called *Lipschitz* if there exists a number $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M \qquad \text{for all } x, y \in X.$$

Show that if $f : X \to \mathbb{R}$ is Lipschitz then it is uniformly continuous on X.

(3) Suppose that f is continuous on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some $k \geq 0$ then f is uniformly continuous on $[0, \infty)$.

(4) Let $f(x) = \sqrt{x}$. Prove that f is uniformly continuous on $[0, \infty)$.
Hint: Prove that f is uniformly continuous on $[1, \infty)$ and use (3).

(5) Suppose that $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} h(x) = L$ and that

$$f(x) \leq g(x) \leq h(x)$$

for all x in (a, b) for some b. Use the ε–δ definition to prove that $\lim_{x \to a^+} g(x) = L$.
Hint: see Exam 1.

(6) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

This function is similar to a function we've seen before; it is continuous at $x = 0$ and discontinuous everywhere else. Prove that f is differentiable at $x = 0$.
So you can work on this problem before we cover derivatives: We say f is differentiable at a if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. *Hint:* I personally find the ε–δ definition easier for this problem.

Extra practice (do not turn these in)

(1) Exercises 19.1, 19.2, 19.5, 19.9, 19.10

(2) Exercises 20.1, 20.2, 20.4, 20.11, 20.16