MATH 131A HOMEWORK 6

(1) Suppose that (a_k) is a sequence with the property that $a_k \in \{0, 1, 2, \ldots, 9\}$ for all $k \geq 1$. Prove that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k}$$

is a convergent series. Write a brief sentence about why this is useful (if you're having trouble with this last part, glance at §16 of your textbook).

(2) (a) (Summation by parts) Let (x_n) and (y_n) be sequences and let (s_n) denote the sequence of partial sums of (x_n); i.e. $s_n = x_1 + \ldots + x_n$. Prove that

$$\sum_{j=m+1}^{n} x_j y_j = s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}).$$

This is the discrete analogue of integration by parts. *Hint:* $x_j = s_j - s_{j-1}$.

(b) (Dirichlet's Test) Suppose that the partial sums s_n above are bounded (but not necessarily convergent). Suppose that (y_n) satisfies $y_1 \geq y_2 \geq y_3 \geq \ldots \geq 0$ and that $\lim y_n = 0$. Prove that $\sum x_n y_n$ converges.
Hint: Let M be an upper bound for (s_n), then use part (a) to show that

$$\left| \sum_{j=m+1}^{n} x_j y_j \right| \leq M \left(y_{n+1} + y_{m+1} + \sum_{j=m+1}^{n} (y_j - y_{j+1}) \right) \leq 2M y_{m+1}.$$

Then apply the Cauchy criterion.
Note: by letting $x_n = (-1)^n$ this gives another proof of the alternating series test.

(3) Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions.
(a) Prove that $\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$.
(b) Suppose that f and g are continuous. Prove that $\min(f, g)$ is continuous.
Hint: see Example 5 on p. 130 in your book.

(4) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Prove that g is continuous at $x = 0$.
(b) Prove that g is not continuous at $x = \dfrac{1}{2}$.
Hint: you may want to use the sequence definition for one of these and the ε-δ definition for the other.

Extra Practice (do not turn these in)

(1) Exercises 15.1, 15.6, and 15.7 on p. 109.

(2) Exercises 17.1 – 17.3, 17.5, 17.6, 17.9, 17.10, and 17.12 on pp. 130–132.