MATH 131A HOMEWORK 4

(1) Suppose that $S \subseteq \mathbb{R}$ is bounded. Prove that $\inf S = -\sup(-S)$ (here $-S$ means the set of all real numbers $-s$ where $s \in S$). *Hint:* let $s_0 = \sup(-S)$. The definition of inf requires two things to be true: $-s_0 \leq s$ for all $s \in S$ and if $t \leq s$ for all $s \in S$ then $t \leq -s_0$. Use the fact that $s_0 = \sup(-S)$ to prove these two statements.

(2) Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$. In other words,

$$A + B := \{a + b : a \in A, b \in B\}.$$

Prove that $\sup(A + B) = \sup A + \sup B$. *(see exercise* 4.14 *on p.* 28 *for a hint)*

(3) Let $a_1 = 1$ and for $n \geq 1$ define $a_{n+1} = \left(1 - \dfrac{1}{(n+1)^2}\right) a_n$.

(a) Prove that (a_n) is convergent. *Hint:* decreasing and bounded below.

(b) Prove by induction that $a_n = \dfrac{n+1}{2n}$.

(c) Find $\lim_{n\to\infty} a_n$. You may use the standard arithmetic of limits, i.e. Theorems 9.2–9.7.

(4) Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n)$.

(a) Prove that (σ_n) is an increasing sequence. *Hint:* First prove that $s_{n+1} \geq \sigma_n$; then consider the quantity $(n + 1)\sigma_{n+1} - n\sigma_n$.

(b) Suppose that (s_n) is bounded. Prove that (σ_n) is a convergent sequence.

Extra Practice (do not turn these in)

(1) Exercises 4.1 – 4.4 (p. 26): practice finding upper and lower bounds for sets, including suprema and infima.

(2) Exercise 4.7 (p. 27): properties of inf and sup for subsets of $\mathbb{R}$.

(3) Exercise 4.12 (p. 27): the irrationals are dense in $\mathbb{R}$.

(4) Exercises 9.1 – 9.3 (p. 54): practice with limits.

(5) Exercises 10.9 – 10.11 (p. 65): practice with recursively defined sequences.