MATH 131A HOMEWORK 3

(1) Suppose that (a_n) is a convergent sequence and for each $n \geq 1$ let $b_n = a_{n+1}$. Prove that (b_n) is also convergent and that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n.$$

(2) Let (a_n) be the sequence recursively defined by $a_1 = \sqrt{2}$ and

$$a_{n+1} = \sqrt{2a_n}.$$

(a) Prove that $a_n \leq 2$ for all $n \geq 1$. *Hint:* use induction.

(b) Prove that $a_{n+1} \geq a_n$ for all $n \geq 1$. *Hint:* use induction and (a).

(c) Soon we'll prove that bounded monotonic sequences converge. So (a_n) converges because it's bounded by above by 2 and increasing. Use (1) to compute

$$\lim_{n\to\infty} a_n.$$

You may also use the fact that $\lim_{n\to\infty} \sqrt{b_n} = \sqrt{\lim_{n\to\infty} b_n}$ if (b_n) is convergent (we'll prove this soon, too).

(3) Let (F_n) be the sequence of Fibonacci numbers defined by $F_1 = F_2 = 1$ and for $n \geq 2$,

$$F_{n+1} = F_n + F_{n-1}.$$

Let $a_n = F_{n+1}/F_n$ denote the sequence of Fibonacci quotients.

(a) Using the definition of F_n, prove that $a_{n+1} = 1 + \dfrac{1}{a_n}$.

(b) Suppose for a moment that (a_n) converges to some number ϕ. Find ϕ using (1).

(c) For the number ϕ you found above, prove that

$$|a_{n+1} - \phi| \leq \left(\frac{2}{3}\right)|a_n - \phi|.$$

Hint: use the recursion for a_{n+1} and the fact that $\phi = 1 + 1/\phi$. Also, verify numerically that $1/\phi < 2/3$.

(d) Prove that

$$|a_{n+1} - \phi| \leq \left(\frac{2}{3}\right)^n |1 - \phi| \leq \left(\frac{2}{3}\right)^{n+1}.$$

(e) Conclude that (a_n) converges to ϕ.