

MATH 131A HOMEWORK 2

- (1) Prove Lemma 4.2 from the course notes: For all $x \in \mathbb{Q}$ we have

$$-|x| \leq x \leq |x|.$$

- (2) Prove Lemma 4.3 from the course notes: For all $x, y \in \mathbb{Q}$ we have

$$|x| \leq |y| \quad \text{if and only if} \quad -|y| \leq x \leq |y|.$$

Note: this requires you to prove two things: first, the left statement implies the right statement, then the other way.

- (3) Prove that the sequence (a_n) defined by $a_n = \frac{n}{n+1}$ is Cauchy.

- (4) Prove that the sequence (a_n) defined by $a_n = \log n$ is not Cauchy. (For this exercise you will need to travel to the future where we know what $\log n$ means. You are welcome to use any properties of $\log$ that you know.)

- (5) Prove Lemma 5.3 from the course notes: If x, y, z are real numbers then (a) $x = x$, (b) $x = y$ implies $y = x$, and (c) $x = y$ and $y = z$ implies $x = z$.

- (6) Prove Lemma 5.17 from the course notes: If $x = \text{"lim"} a_n$ then $|x| = \text{"lim"} |a_n|$.

- (7) The real number which has decimal expansion $.999\dots$ is represented by the sequence (a_n) where $a_n = 1 - 10^{-n}$. That is, $.999\dots = \text{"lim"} a_n$. Prove that $.999\dots = 1$.

- (8) (a) Suppose x is a positive real number. Prove that there exists a natural number N such that $0 < 1/N < x$. *Hint:* x^{-1} is represented by a Cauchy sequence.
(b) Suppose that x, y are real numbers such that $y - x > 1$. Prove that there exists an integer m such that $x < m < y$. *Hint:* Suppose not and find a contradiction.
(c) Given any two real numbers x and y with $x < y$, prove that there is a rational number $\frac{m}{N}$ such that $x < \frac{m}{N} < y$. *Hint:* use (a) and (b).