MATH 131A HOMEWORK 10

(1) Let $k \in \mathbb{R}$. Show, directly from the definition, that the function $f(x) = k$ is integrable on any interval $[a, b]$ and compute $\int_a^b f$.

(2) Define $f : [0, 2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Show that f is integrable on $[0, 2]$. *Hint:* given $\varepsilon > 0$, construct a partition P such that $U(f, P) - L(f, P) < \varepsilon$.

(3) Suppose that $f : [0, 2] \to \mathbb{R}$ is an integrable function and that $g : [0, 2] \to \mathbb{R}$ satisfies

$$g(x) = f(x) \quad \text{if } x \neq 1, \quad g(x) \neq f(x) \quad \text{if } x = 1.$$

Prove that g is integrable and that $\int_0^2 g = \int_0^2 f$. *Hint:* consider the function $h = f - g$ and mimic your proof of (2).

(4) Prove that the function

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 5 & \text{if } x = 0, \end{cases}$$

is integrable on $[0, 1]$. *Hint:* Given $\varepsilon > 0$, break the interval $[0, 1]$ into two parts: $[0, a] \cup [a, 1]$, for some a that depends on ε (I don't think that $a = \varepsilon/2$ works, but it will be something like that). Use the fact that the first interval is very short, and the fact that f is continuous on the second interval, to prove that there exists a partition P such that $U(f, P) - L(f, P) < \varepsilon$.

(5) Use the Fundamental Theorem of Calculus II to give a simple proof of the Fundamental Theorem of Calculus I in the case that g' is continuous (see Exercise 34.1).

(6) One way to define the function $\log x$ is as an integral. For $x > 0$ let

$$L(x) = \int_1^x \frac{dt}{t}.$$

(a) Compute $L(1)$ and $L'(x)$.
(b) Prove that $L(x)$ is strictly increasing for $x \geq 1$.
(c) Prove that $L(xy) = L(x) + L(y)$ for every $x, y > 0$. *Hint:* thinking of y as a constant, use (a) and the chain rule to show that $[L(xy)]' = L'(x)$, so $L(xy)$ and $L(x)$ differ by a constant. Now use (a) to compute the constant.

Extra practice (do not turn these in)

1. Exercises 32.6, 32.7, 32.8
2. Exercises 33.4, 33.13
3. Exercises 34.2, 34.3, 34.4, 34.11, 34.12
4. Suppose that f and g are continuous and that $g(x) \geq 0$ for all $x \in [a, b]$. Prove that there exists a $c \in (a, b)$ such that
$$\int_a^b fg = f(c) \int_a^b g.$$
Hint: If $\int_a^b g = 0$ then this is pretty straightforward. If $\int_a^b g \neq 0$ then mimic the proof of the Intermediate Value theorem for Integrals to prove that there is a $c \in (a, b)$ such that
$$f(c) = \frac{1}{\int_a^b g} \int_a^b fg.$$