MATH 131A HOMEWORK 1

(0) (nothing to turn in for this exercise) Read the first three sections (through p. 15) of the following handout on writing mathematical proofs (clickable link):

`https://math.berkeley.edu/~hutching/teach/proofs.pdf`.

Also read the first four pages (up to "Writing Mathematical Formulas") of this handout on mathematical writing:

`https://sites.math.washington.edu/~lee/Writing/writing-proofs.pdf`.

I realize that it's a lot to read, so you can do this over the first two weeks. There are several good points in these handouts which we simply won't be able to cover in much detail in this course. Return to these as references whenever you have questions about writing proofs. This information will help you in every subsequent math course you take (and beyond).

(1) (For this exercise only, you may time travel to the future, where we are very familiar with algebraic mainuplation and inequalities; you do not need to state which properties you are using at each step. See the proof of Proposition 2.3 of the course notes for an example of what I'm looking for.)

(a) Prove: for all natural numbers n, we have

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(b) Prove: for all natural numbers $n \geq 4$ we have $n! > n^2$.
Notes: if you have forgotten, $n! := n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$. Also, the principle of mathematical induction can be extended as follows: let m be a fixed integer (for example, $m = 4$). Suppose that P_m is true, and that whenever P_n is true ($n \geq m$), P_{n+1} is true. Then P_n is true for all $n \geq m$. (i.e. induction doesn't have to start at P_1; you can start at P_4, for instance.)

(2) Prove (4) from Proposition 3.1 in the course notes: if $a, b \in \mathbb{Z}$ then

$$(-a) \cdot (-b) = a \cdot b.$$

At each step, state which property you are using (you are welcome to use (1), (2), and (3) from Proposition 3.1 as well).
Hint: Show that $(-a) \cdot (-b) + -(a \cdot b) = a \cdot b + -(a \cdot b)$ and use (1).

(3) Prove (6) from Proposition 3.1 in the course notes: if $a, b, c \in \mathbb{Z}$ and $a \cdot c = b \cdot c$ with $c \neq 0$ then $a = b$. At each step, state which property you are using (you are welcome to use (1) – (5) from Proposition 3.1 as well).
Note: the statement is for integers, not rational numbers, so you may not multiply by

c^{-1}, as it is unlikely that c has a multiplicative inverse in $\mathbb{Z}$.
Hint: Consider the quantity $a \cdot c - b \cdot c$.

(4) Using the field properties from Section 3 of the course notes (commutative law, associative law, etc.) prove that
$$(a + b)^2 = a^2 + 2ab + b^2.$$
At each step, state which property you are using.

(5) Prove (5) from Proposition 3.2 of the course notes: $0 < 1$.
Hint: $1^2 = 1$.

(6) Prove (7) from Proposition 3.2 of the course notes: if $0 < a < b$ then $0 < b^{-1} < a^{-1}$.
Hint: Use (6) and (O5).