MATH 131A: REAL ANALYSIS (BIG IDEAS)

Theorem 1 (The Triangle Inequality)**.** *For all $x, y \in \mathbb{R}$ we have*

$$|x + y| \leq |x| + |y|.$$

Proposition 2 (The Archimedean property)**.** *For each $x \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ such that $n > x$.*

Definition 3. A *Cauchy sequence* is a sequence of real numbers (a_n) such that for every $\varepsilon > 0$ there exists a positive integer N such that

$$|a_m - a_n| < \varepsilon \quad \text{whenever} \quad m, n \geq N.$$

Proposition 4. *For each $n \geq 1$ let $H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Then the sequence $(H_1, H_2, H_3, \ldots)$ is not a Cauchy sequence.*

Proof. For each $n \in \mathbb{N}$, consider the difference

$$|H_{2n} - H_n| = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}.$$

Notice that there are exactly n terms in the sum, and each term is larger than (or equal to) the smallest one $\frac{1}{2n}$. Thus

$$|H_{2n} - H_n| \geq \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.$$

So for each $N \geq 1$ let $m = 2N$ and $n = N$; then $|H_m - H_n| \geq \frac{1}{2}$. □

Definition 5. A sequence (a_n) is *bounded* if there exists M such that $|a_n| \leq M$ for all $n \geq 1$.

Proposition 6. *Every Cauchy sequence is bounded.*

Proof. Suppose that (a_n) is a Cauchy sequence. Taking $\varepsilon = 1$, this means that there is some $N \in \mathbb{N}$ for which $|a_m - a_n| \leq 1$ for all $m, n \geq N$. This splits our sequence into a finite piece (indices 1 through N) and an infinite piece (index N and after). By Lemma **??**, the finite piece is bounded, say by M. Using the Triangle Inequality we find that

$$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| < 1 + M.$$

It follows that (a_n) is bounded by $M + 1$. □

Definition 7. Two sequences (a_n) and (b_n) are *equivalent* if for each $\varepsilon > 0$ there exists a natural number N (which can depend on ε) such that

$$|a_n - b_n| < \varepsilon \quad \text{whenever} \quad n \geq N.$$

Proposition 8 ($\mathbb{Q}$ is dense in $\mathbb{R}$)**.** *For every pair of real numbers x, y with $x < y$ there exists a rational number q such that*

$$x < q < y.$$

Definition 9. Let (a_n) be a sequence of real numbers. We say that (a_n) converges to $L \in \mathbb{R}$ (and we write $\lim\limits_{n\to\infty} a_n = L$) if for every $\varepsilon > 0$ there exists a natural number N for which

$$|a_n - L| < \varepsilon \quad \text{for all} \quad n \geq N.$$

Theorem 10 ($\mathbb{R}$ is complete)**.** *Every Cauchy sequence in $\mathbb{R}$ converges to a real number.*

Definition 11. Suppose that $E \subseteq \mathbb{R}$ is a set of real numbers and let $M, m \in \mathbb{R}$. We say that M is a *least upper bound* for E if (a) M is an upper bound for E and (b) if M' is another upper bound for E, then $M' \geq M$. Similarly, we say that m is a *greatest lower bound* for E if (a) m is a lower bound for E and (b) if m' is another lower bound for E then $m' \leq m$.

Theorem 12 (Least upper bound property)**.** *Let E be a non-empty subset of $\mathbb{R}$. If E has an upper bound, then it has a least upper bound.*

Proof. Since E is non-empty, there is some element $a_1 \in E$, and since E has an upper bound, we can choose some upper bound b_1. Clearly $a_1 \leq b_1$. We will define two sequences (a_i) and (b_i) inductively such that $a_n \leq b_n$ for all n.

For each $n \geq 1$, suppose that we have constructed a_n and b_n such that $a_n \leq b_n$. Let $K = \frac{a_n+b_n}{2}$ be the average of a_n and b_n. Note that $a_n \leq K \leq b_n$. If K is an upper bound for E, let $a_{n+1} = a_n$ and let $b_{n+1} = K = \frac{a_n+b_n}{2}$. Then $|b_{n+1} - a_{n+1}| = |\frac{a_n+b_n}{2} - a_n| = \frac{b_n-a_n}{2}$. If K is not an upper bound for E, then there is some element $x \in E$ such that $x > K$ (otherwise K would be an upper bound). In that case, let $a_{n+1} = x$ and let $b_{n+1} = b_n$. Then $|b_{n+1} - a_{n+1}| = b_n - x < b_n - K = \frac{b_n-a_n}{2}$. Note that in either case $|b_{n+1} - a_{n+1}| \leq \frac{b_n-a_n}{2}$. Also in either case a_n is increasing and b_n is decreasing.

Since $a_n \in E$ for all n and b_n is an upper bound for E for all n, we have

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq b_3 \leq b_2 \leq b_1.$$

Using the inequality $|b_{n+1} - a_{n+1}| \leq \frac{b_n-a_n}{2}$ one can show by induction that

$$|b_{n+1} - a_{n+1}| \leq \frac{b_1 - a_1}{2^n}. \tag{1}$$

We will show that the sequences (a_n) and (b_n) are Cauchy.

Let $\varepsilon > 0$ be given. Choose $N \geq 1$ such that $2^N > \frac{b_1-a_1}{\varepsilon}$. Let $m, n \geq N$. Without loss of generality, assume that $m \geq n$. Then $a_m \geq a_n$, and we have

$$|a_m - a_n| = a_m - a_n \leq b_n - a_n \leq \frac{b_1 - a_1}{2^n} < \varepsilon.$$

Thus (a_n) is Cauchy. A very similar argument shows that (b_n) is Cauchy.

Using (1) one can show that (a_n) and (b_n) are equivalent sequences. Therefore they have the same limit, say L. I claim that L is the least upper bound of E. On the one hand, if $x \in E$ then $x \leq b_n$ for all n since every b_n is an upper bound for E. Thus $x \leq L$. Since x was an arbitrary element of E, we conclude that L is an upper bound for E. On the other hand, if L' is any other upper bound, then $L' \geq a_n$ for all n since every a_n is an element of E. Thus $L' \geq L$. It follows that L is the least upper bound of E. □

Proposition 13 (Bounded monotonic sequences are convergent)**.** *If a sequence is increasing and bounded above then it converges.*

Proof. Suppose that (a_n) is increasing and bounded above. Let $E = \{a_n : n \geq 1\}$. Then (a_n) has a least upper bound which we will call $s = \sup a_n$. We will show that $\lim a_n = s$.

Let $\varepsilon > 0$ be given. Since s is the least upper bound, $s - \varepsilon$ is not an upper bound for E. So there exists some $N \geq 1$ such that $a_N > s - \varepsilon$. Since (a_n) is increasing, it follows that $a_n \geq a_N > s - \varepsilon$ for all $n \geq N$. Thus, for all $n \geq N$ we have

$$|a_n - s| = s - a_n \leq s - a_N < s - (s - \varepsilon) = \varepsilon.$$

Therefore (a_n) converges to s. $\square$

Definition 14. A *subsequence* of a sequence (a_n) is a sequence of the form (a_{n_k}), where n_k is a strictly increasing sequence of natural numbers.

Theorem 15 (Bolzano-Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

Definition 16. Let (a_n) be a sequence. Then

$$\limsup a_n = \lim_{N\to\infty} \sup\{a_n : n > N\},$$
$$\liminf a_n = \lim_{N\to\infty} \inf\{a_n : n > N\}.$$

Definition 17. Let (a_k) be a sequence, and for each $n \geq 1$ let

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

be the nth partial sum. We say that $\sum a_k$ *converges* if the sequence (s_n) is a convergent sequence. If $\sum |a_n|$ is convergent, we say the series $\sum a_n$ is *absolutely convergent.*

Proposition 18 (Cauchy criterion for series)**.** *A series $\sum a_n$ is convergent if and only if for each $\varepsilon > 0$ there exists a number $N \geq 1$ such that*

$$\left|\sum_{k=m}^{n} a_k\right| < \varepsilon$$

whenever $n \geq m \geq N$.

Corollary 19. *If $\sum a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.*

Proof. Apply the Cauchy criterion with $m = n$ to get: for every $\varepsilon > 0$ there exists a number $N \geq 1$ such that $|a_n| < \varepsilon$ whenever $n \geq N$. $\square$

Proposition 20 (The comparison test)**.** *Let $\sum a_n$ be a series of nonnegative terms.*

(1) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n then $\sum b_n$ converges.
(2) If $\sum a_n = +\infty$ and $b_n \geq a_n$ for all n then $\sum b_n$ diverges.

Proof. (1) Suppose that $\sum a_n$ converges. Then it satisfies the Cauchy criterion. Given $\varepsilon > 0$ there exists a number $N \geq 1$ such that

$$\sum_{k=n}^{m} a_k = \left|\sum_{k=n}^{m} a_k\right| < \varepsilon$$

for all $n \geq m \geq N$ (note $a_k \geq 0$ for all k). So for $m \geq n \geq N$ we have

$$\left|\sum_{k=n}^{m} b_k\right| \leq \sum_{k=n}^{m} |b_k| \leq \sum_{k=n}^{m} a_k < \varepsilon,$$

where we used the Triangle inequality in the first step. Thus $\sum b_n$ satisfies the Cauchy criterion, and therefore it converges.

(2) Let (s_n) be the sequence of partial sums for $\sum a_n$ and let (t_n) be the sequence of partial sums for $\sum b_n$. Then $t_n \geq s_n$ for all n. Since $\lim s_n = +\infty$ we have $\lim t_n = +\infty$ too. $\square$

Proposition 21 (Alternating series test)**.** *If (a_n) is a decreasing sequence of nonnegative real numbers and $\lim a_n = 0$ then the alternating series $\sum(-1)^{n+1}a_n$ converges.*

Proof. Let (s_n) denote the sequence of partial sums. Let (e_n) be the subsequence of even-indexed terms and (o_n) the subseqeunce of odd-indexed terms. Then $o_n = s_{2n-1}$ and $e_n = s_{2n}$:

$$(s_n) = o_1, e_1, o_2, e_2, o_3, e_3, \ldots.$$

The even-indexed terms are increasing since

$$e_{n+1} - e_n = s_{2n+2} - s_{2n} = -a_{2n+2} + a_{2n+1} \geq 0.$$

Similarly, the odd-indexed terms are decreasing since

$$o_{n+1} - o_n = s_{2n+1} - s_{2n-1} = a_{2n+1} - a_{2n} \leq 0.$$

I claim that every even-indexed term is less than or equal to every odd-indexed term:

$$e_m \leq o_n \quad \text{for all } m, n \in \mathbb{N}.$$

First, $e_n \leq o_{n+1} \leq o_n$ because $o_{n+1} - e_n = s_{2n+1} - s_{2n} = a_{2n+1} \geq 0$ and (o_n) is decreasing. Now, if $m \leq n$, then because (e_n) is increasing, $e_m \leq e_n \leq o_n$. On the other hand, if $m \geq n$ then $o_n \geq o_m \geq e_m$. To summarize,

$$e_1 \leq e_2 \leq e_3 \leq \cdots \leq o_3 \leq o_2 \leq o_1.$$

Since (e_n) is increasing and bounded above, it converges, say to L_e. Similarly, (o_n) converges, say to L_o. But

$$L_o - L_e = \lim_{n\to\infty} o_{n+1} - \lim_{n\to\infty} e_n = \lim_{n\to\infty}(o_{n+1} - e_n) = \lim_{n\to\infty} a_{2n+1} = 0.$$

Thus $L_o = L_e$. It follows that (s_n) converges. □

Definition 22. Let $X \subseteq \mathbb{R}$. A function $f : X \to \mathbb{R}$ is *continuous at the point* $a \in X$ if, for every sequence (a_n) (with $a_n \in X$) converging to a, the sequence $(f(a_n))$ converges to $f(a)$. Put another way, f is continuous at a if

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

for every sequence (a_n) that converges to a.

Definition 23. Let $X \subseteq \mathbb{R}$. A function $f : X \to \mathbb{R}$ is *continuous at the point* $a \in X$ if, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{whenever} \quad x \in X \text{ and } |x - a| < \delta.$$

Theorem 24 (Extreme Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded (i.e. there exists a real number M such that $|f(x)| \leq M$ for all $x \in [a, b]$). Furthermore, f attains its maximum and minimum values on $[a, b]$ (i.e. there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$).*

Proof. We argue by contradiction. Suppose that f is not bounded on $[a, b]$. Then for each $n \in \mathbb{N}$ there exists a number $x_n \in [a, b]$ such that $|f(x_n)| > n$. So we get a sequence (x_n) which is bounded (it's always inside $[a, b]$). By the Bolzano-Weierstrass theorem, there exists a convergent subsequence (x_{n_k}) of (x_n), say that $\lim x_{n_k} = L$. Then $L \in [a, b]$, so by assumption f is continuous at L. So we have a sequence (x_{n_k}) which converges to L but the sequence $(f(x_{n_k}))$ doesn't converge to $f(L)$ (since $|f(x_{n_k})| \to +\infty$). This contradicts the continuity of f, so we must have that f is bounded.

Now let $M := \sup\{f(x) : x \in [a,b]\}$. We just proved that M is a finite number. For each $n \in \mathbb{N}$ there exists a number $y_n \in [a,b]$ such that $M - \frac{1}{n} \leq f(y_n) \leq M$ (otherwise M wouldn't be the sup). By the squeeze theorem, $\lim f(y_n) = M$. We aren't guaranteed that (y_n) is a convergent sequence, but it is bounded (always in $[a,b]$) so Bolzano-Weierstrass gives us a convergent subsequence (y_{n_k}). Say $\lim y_{n_k} = y_0$. Since f is continuous at y_0 we have

$$f(y_0) = f(\lim y_{n_k}) = \lim f(y_{n_k}) = \lim f(y_n) = M.$$

(In the third equality we used that if a sequence is convergent, then every subsequence converges to the same limit.) So f attains its maximum M. The assertion for the minimum is similar. □

Theorem 25 (Intermediate Value Theorem)**.** *Let $I \subseteq \mathbb{R}$ be an interval and suppose that $f : I \to \mathbb{R}$ is continuous. Let $a, b \in I$ such that $a < b$. Then for every y between $f(a)$ and $f(b)$ (i.e. either $f(a) < y < f(b)$ or $f(a) > y > f(b)$) there exists at least one $x \in (a,b)$ such that $f(x) = y$.*

Proof. Assume that $f(a) < y < f(b)$ (the other case is similar). We will construct two sequences (a_n) and (b_n) which converge to x_0 such that $f(a_n) \geq y$ and $f(b_n) \leq y$.

Let $S = \{x \in [a,b] : f(x) < y\}$. Then S is not empty because $a \in S$. Since S is also bounded above (by b) it has a least upper bound, say $x_0 = \sup S$. For each $n \in \mathbb{N}$, the number $x_0 - \frac{1}{n}$ is not an upper bound for S so there exists $b_n \in S$ such that $x_0 - \frac{1}{n} < b_n \leq x_0$. By the squeeze theorem, $\lim b_n = x_0$. Since f is continuous and since $f(b_n) < y$ we have

$$f(x_0) = f(\lim b_n) = \lim f(b_n) \leq y.$$

Now for each $n \in \mathbb{N}$ let $a_n = \min\{b, x_0 + \frac{1}{n}\}$. Then for each n, $a_n \notin S$ so $f(a_n) \geq y$. Again by the squeeze theorem, $\lim a_n = x_0$ so we have

$$f(x_0) = f(\lim a_n) = \lim f(a_n) \geq y.$$

Thus $f(x_0) = y$. □

Definition 26. A function $f : X \to \mathbb{R}$ is *uniformly continuous* on X if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad x, y \in X \text{ and } |x - y| < \delta.$$

Proposition 27. *If $f : X \to \mathbb{R}$ is uniformly continuous on X and if (a_n) is a Cauchy sequence then $(f(a_n))$ is a Cauchy sequence.*

Proof. Let (a_n) be a Cauchy sequence and let $\varepsilon > 0$. Since f is uniformly continuous on X, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $x, y \in X$ and $|x - y| < \delta$. Since (a_n) is Cauchy, there exists and $N \geq 1$ such that

$$|a_m - a_n| < \delta$$

for all $m, n \geq N$. Thus, for such m, n we have

$$|f(a_m) - f(a_n)| < \varepsilon$$

so $(f(a_n))$ is a Cauchy sequence. □

Proposition 28. *A function $f : (a,b) \to \mathbb{R}$ can be extended to a continuous function on $[a,b]$ if and only if it is uniformly continuous on (a,b).*

Proof. ($\Rightarrow$) Suppose that f can be extended to a continuous function $\tilde{f} : [a, b] \to \mathbb{R}$. Then by Proposition **??** $\tilde{f}$ is uniformly continuous on $[a, b]$. It follows immediately that f is uniformly continuous on (a, b).

($\Leftarrow$) Suppose that f is uniformly continuous on (a, b). We must decide how to define $\tilde{f}(a)$ and $\tilde{f}(b)$ to make $\tilde{f} : [a, b] \to \mathbb{R}$ continuous. It suffices to show how to do this for $\tilde{f}(a)$ (since $\tilde{f}(b)$ is similar). We define

$$\tilde{f}(a) = \lim_{n\to\infty} f(a_n) \text{ for any sequence } (a_n) \text{ in } (a, b) \text{ converging to } a. \tag{2}$$

Claim 1: if (a_n) is a sequence in (a, b) converging to a then the sequence $(f(a_n))$ converges. To prove this, note that (a_n) is a Cauchy sequence, so by Proposition 27, $(f(a_n))$ is Cauchy, so it converges.

Claim 2: if (a_n) and (a'_n) both converge to a then $\lim f(a_n) = \lim f(a'_n)$. To prove this, form the sequence (z_n) via

$$(z_n) = (a_1, a'_1, a_2, a'_2, a_3, a'_3, \ldots).$$

It should be clear that $\lim z_n = \lim a_n = \lim a'_n = a$ so by Claim 1, $\lim f(z_n)$ exists. But now $(f(a_n))$ and $(f(a'_n))$ are subsequences of the convergent sequence $(f(z_n))$ so they converge to the same limit, i.e. $\lim f(a_n) = \lim f(a'_n)$.

Therefore (2) makes sense, and it is clear by construction that $\tilde{f}$ is continuous at a. $\square$

Definition 29. Suppose that X contains the interval (c, b) for some $c < a < b$. We say that $\lim\limits_{x\to a} f(x) = L$ if

$$\lim_{n\to\infty} f(a_n) = L \qquad \text{for every sequence } (a_n) \text{ in } (c, a) \cup (a, b) \text{ converging to } a.$$

Definition 30. Let f be a function defined on the set $(c, a) \cup (a, b)$ for some $c < a < b$. Then $\lim\limits_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever } 0 < |x - a| < \delta.$$

Definition 31. Let $f : X \to \mathbb{R}$ be a function and let $a \in X$ such that there is an interval $(c, b) \subseteq X$ containing a. We say that f is *differentiable* at a if the limit

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite; we call this limit *the derivative of f at a*, and write it $f'(a)$.

Proposition 32 (Differentiability implies continuity)**.** *Suppose that f is defined on (c, b) for some $c < a < b$. If f is differentiable at a then f is continuous at a.*

Proof. Recall that when thinking about the limit of a function at a we consider points x close to a but not equal to a. So we are justified in writing

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left[(x - a)\frac{f(x) - f(a)}{x - a} + f(a)\right] = f(a) + \lim_{x\to a}(x - a)\frac{f(x) - f(a)}{x - a}.$$

Using our assumption that f is differentiable at a, the limit of the quotient on the right-hand side above exists and is finite. So we have

$$\lim_{x\to a}(x - a)\frac{f(x) - f(a)}{x - a} = \lim_{x\to a}(x - a) \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = 0 \cdot [\text{something finite}] = 0.$$

This shows that $\lim\limits_{x\to a} f(x) = f(a)$, as desired. $\square$

Proposition 33 (Product rule)**.** *Let f and g be functions that are differentiable at a. Then fg is differentiable at a and*

$$(fg)'(a) = f(a)g'(a) + f'(a)g(a).$$

Proof. Adding zero in a clever way, we compute

$$\begin{aligned}\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a} &= \lim_{x\to a}\frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a}\\ &= \lim_{x\to a} f(x)\frac{g(x)-g(a)}{x-a} + g(a)\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\ &= f(a)g'(a)+f'(a)g(a),\end{aligned}$$

as desired. □

Theorem 34 (Rolle's Theorem)**.** *Suppose that f is continuous on $[a,b]$ and differentiable on (a,b) and that $f(a)=f(b)$. Then there exists a point $c\in(a,b)$ such that $f'(c)=0$.*

Proof. By the Extreme Value Theorem, since f is continuous on the closed interval $[a,b]$ it attains its maximum and minimum on $[a,b]$. So there exist $y,z\in[a,b]$ such that $f(y)\le f(x)\le f(z)$ for all $x\in[a,b]$. Suppose that z is an interior point, i.e. $z\in(a,b)$. Let (z_n) be a sequence in $(a,z)\cup(z,b)$ converging to z and let (z_n^+) and (z_n^-) be the subsequences of (z_n) such that $z_n^+>z$ and $z_n^-<z$. Then (since $f(z)$ is a maximum)

$$\lim_{n\to\infty}\frac{f(z_n^+)-f(z)}{z_n^+-z}\le 0 \quad\text{and}\quad \lim_{n\to\infty}\frac{f(z_n^-)-f(z)}{z_n^--z}\ge 0.$$

(Both limits exist because f is differentiable at z.) Since subsequences of a convergent sequence converge to the same limit, we see that

$$\lim_{n\to\infty}\frac{f(z_n)-f(z)}{z_n-z}\le 0 \quad\text{and}\quad \lim_{n\to\infty}\frac{f(z_n)-f(z)}{z_n-z}\ge 0.$$

It follows that $f'(z)=0$. By the same reasoning, we have $f'(y)=0$ if $y\in(a,b)$. The only case left is if y and z are both endpoints of $[a,b]$. But since $f(a)=f(b)$ this forces $f(y)=f(x)=f(z)$ for all $x\in[a,b]$, so $f'(x)=0$ for all $x\in(a,b)$. □

Theorem 35 (The Mean Value Theorem)**.** *Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $c\in(a,b)$ such that*

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Proof. Define

$$g(x)=f(x)-\left(\frac{f(b)-f(a)}{b-a}(x-b)+f(b)\right).$$

The function in parentheses is just the equation of the secant line connecting $(a,f(a))$ and $(b,f(b))$. We compute

$$g(a)=0 \quad\text{and}\quad g(b)=0.$$

Clearly g is continuous on $[a,b]$ and differentiable on (a,b), so Rolle's theorem applies. So there is some point $c\in(a,b)$ such that $g'(c)=0$. On the other hand,

$$g'(c)=f'(c)-\frac{f(b)-f(a)}{b-a}.$$

This proves the theorem. □

Definition 36. Suppose that f is bounded on $[a,b]$. A *partition* P of $[a,b]$ is a finite ordered set

$$P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}.$$

The intervals $[x_{k-1}, x_k]$ are called *subintervals* of P. For each subinterval, let

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\},$$
$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

The *lower Darboux sum* $L(f,P)$ of f with respect to P is

$$L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

and the *upper Darboux sum* $U(f,P)$ of f with respect to P is

$$U(f,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

We define

$$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$$

and

$$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\},$$

and we say that f is *integrable* on $[a,b]$ if $L(f) = U(f)$. In that case we write

$$\int_a^b f = L(f) = U(f).$$

Proposition 37 (Cauchy criterion for integrals)**.** *Suppose that f is bounded on $[a,b]$. Then f is integrable if and only if for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that*

$$U(f,P) - L(f,P) < \varepsilon.$$

Proof. ($\Leftarrow$) Suppose that such a partition P exists for each $\varepsilon > 0$. Then

$$U(f) - L(f) \le U(f,P) - L(f,P) < \varepsilon.$$

Since ε is arbitrary, this implies that $U(f) = L(f)$.

($\Rightarrow$) Now suppose that f is integrable. Let $\varepsilon > 0$ be given. Since $U(f)$ is the greatest lower bound of the $U(f,P)$ over all partitions P, the number $U(f) + \frac{\varepsilon}{2}$ is no longer a lower bound, so there exists a partition P_1 such that

$$U(f,P_1) < U(f) + \frac{\varepsilon}{2}.$$

Similarly, there exists a partition P_2 such that

$$L(f,P_2) > L(f) - \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$. Since integrability means that $U(f) = L(f)$, we have

$$\begin{aligned} U(f,P) - L(f,P) &\le U(f,P_1) - L(f,P_2) \\ &= U(f,P_1) - U(f) + L(f) - L(f,P_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This completes the proof. $\square$

Proposition 38. *Every continuous function on $[a,b]$ is integrable.*

Proof. First, we observe that because f is continuous on the closed interval $[a,b]$ it is uniformly continuous on $[a,b]$. So, given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad \text{whenever } |x-y| < \delta.$$

Let P be a partition of $[a,b]$ such that $x_k - x_{k-1} < \delta$ for every subinterval. By the extreme value theorem applied to $[x_{k-1}, x_k]$, the function f attains its maximum M_k and its minimum m_k for some points z_k, y_k in the interval $[x_{k-1}, x_k]$. And whatever those points z_k, y_k are, they are separated by a distance of at most δ. So

$$M_k - m_k = f(z_k) - f(y_k) < \frac{\varepsilon}{b-a}.$$

It follows that

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \frac{\varepsilon}{b-a} \sum_{k=1}^{n} (x_k - x_{k-1}) = \varepsilon$$

since the last sum is telescoping and equals $b-a$. By the Cauchy criterion, f is integrable on $[a,b]$. □

Theorem 39 (The Fundamental Theorem of Calculus I)**.** *If g is continuous on $[a,b]$ and differentiable on (a,b), and if g' is integrable on (a,b) then*

$$\int_a^b g' = g(b) - g(a).$$

Proof. Let P be a partition of $[a,b]$. For each k, apply the Mean Value Theorem to the interval $[x_{k-1}, x_k]$ to find a point $c_k \in (x_{k-1}, x_k)$ such that $(x_k - x_{k-1})g'(c_k) = g(x_k) - g(x_{k-1})$. Then

$$g(b) - g(a) = \sum_{k=1}^{n} (g(x_k) - g(x_{k-1})) = \sum_{k=1}^{n} g'(c_k)(x_k - x_{k-1}).$$

For each k we clearly have

$$\inf\{g'(x) : x \in [x_{k-1}, x_k]\} \leq g'(c_k) \leq \sup\{g'(x) : x \in [x_{k-1}, x_k]\}.$$

It follows that

$$L(g', P) \leq g(b) - g(a) \leq U(g', P).$$

Since this inequality holds for every partition P (and the middle quantity does not depend on P), we have

$$L(g') \leq g(b) - g(a) \leq U(g').$$

But $U(g') = L(g')$ since g' is integrable, so $\int_a^b g' = U(g') = g(b) - g(a)$. □

Theorem 40 (The Fundamental Theorem of Calculus II)**.** *Suppose that f is integrable on $[a,b]$ and define*

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is continuous on $[a,b]$. If f is continuous at $c \in (a,b)$ then F is differentiable at c and

$$F'(c) = f(c).$$

Proof. Let M be such that $|g(x)| \leq M$ for all $x \in [a, b]$. For $x, y \in [a, b]$ we have

$$|F(y) - F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right| = \left| \int_x^y f(t)\,dt \right| \leq \int_x^y |f(t)|\,dt \leq M|x - y|.$$

Therefore F is Lipschitz, so it is uniformly continuous on $[a, b]$.

Suppose that f is continuous at $c \in (a, b)$. Observe that

$$f(c) = \frac{1}{x - c} \int_c^x f(c)\,dt$$

and

$$\frac{F(x) - F(c)}{x - c} = \frac{1}{x - c} \int_c^x f(t)\,dt,$$

so

$$\frac{F(x) - F(c)}{x - c} - f(c) = \frac{1}{x - c} \int_c^x (f(t) - f(c))\,dt.$$

Let $\varepsilon > 0$ be given. Since f is continuous at c, there exists a $\delta > 0$ such that $|f(t) - f(c)| < \varepsilon$ whenever $|t - c| < \delta$. For the integral above we only care about values of t between c and x, and if $|x - c| < \delta$ then certainly $|t - c| < \delta$. For such x it follows that

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| \leq \frac{1}{|x - c|} \int_c^x |f(t) - f(c)|\,dt < \frac{1}{|x - c|} \int_c^x \varepsilon\,dt = \varepsilon.$$

Therefore $F'(c) = f(c)$. □